DRAFT 0.00: TOPOLOGICAL QUANTUM FIELD THEORY

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Abstract. These unedited notes are based on a course offered by Justin Roberts in Winter 2025 at UCSD. They are merely a transcript of what Justin spoke ± some of my own mistakes in transcribing and understanding the content − a lot of references.

Contents

1. Introduction

Over the last 40 years, mathematics—particularly geometry and topology—has been profoundly influenced by ideas originating in physics, specifically quantum field theory. One of the most significant contributions is Topological Quantum Field Theory (TQFT).

An $(n+1)$-dimensional TQFT is a tensor functor from the category of $(n+1)$-dimensional cobordisms to the category of vector spaces. In simple terms, it assigns:

(1) To each closed n-manifold a vector space.
(2) To each $n+1$-manifold with boundary split into input and output, a linear map between these vector spaces.

Intuitively, the n-manifolds represent spacelike slices of spacetime, while the $(n+1)$-manifolds represent spacetime slabs defining time-evolution operators. The tensor product structure

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of state spaces corresponds to the essential quantum-mechanical principle of *locality.* In contrast to physical quantum field theories—where manifolds have metric structures and state spaces are typically infinite-dimensional—a TQFT is metric-independent, yielding finite-dimensional state spaces and topological invariants of closed $(n+1)$-manifolds.

The roots of TQFT trace back to Donaldson's groundbreaking work on 4-manifolds in 1982 [Don83] and Jones' discovery of his polynomial knot invariant in 1984 [Jon85]. However, it was Witten who, in 1988–89, introduced the term TQFT and provided a physical interpretation and generalization of these results [Wit88, Wit89]. In particular, he explained how Jones-type invariants extend to *quantum Chern-Simons invariants* for 3-manifolds. Although Witten's approach relies on physics, including the *Feynman path integral* (which lacks mathematical rigor), the invariants he proposed can be studied rigorously using approaches such as those of Reshetikhin and Turaev.

From a topological perspective, TQFT can be viewed as an axiomatic characterization of a class of manifold invariants, analogous to the Eilenberg-Steenrod axioms for homology but inherently multiplicative in nature.

In this course, we will delve deeper into these ideas, focusing on the (2+1)-dimensional *Chern-Simons theory* and exploring its connections to other mathematical areas. However, we will *not* cover:

(1) Four-manifold invariants and Heegaard–Floer homology.
(2) Higher categories, especially Jacob Lurie's work on TQFT, which requires substantial background knowledge. (We might briefly touch on his work towards the end of the course.)

Prerequisites will be kept minimal, with concise summaries of any necessary background material provided throughout. Some foundational material might also be covered in Math 259.

1.1. **Physical prologue.** In physics, a fundamental concept is the time evolution of a system:

(1) In classical mechanics, this involves studying the evolution of a point in phase space using Hamilton's equations.
(2) In quantum mechanics, states evolve within a Hilbert space via the Schrödinger equation.

Before delving into quantum field theory, consider a Lorentzian manifold W^{n+1} that describes the spacetime evolution from an initial slice M_0^n to a final slice M_1^n (Figure 1). A quantum field theory assigns:

(1) *State spaces* $\mathcal{H}(M_0^n)$ and $\mathcal{H}(M_1^n)$, describing the states of systems on M_0^n and on M_1^n, respectively.
(2) An *evolution operator* $Z(W^{n+1}) : \mathcal{H}(M_0^n) \to \mathcal{H}(M_1^n)$.

Example 1.1. Usually, physicists consider W^{n+1} to be the space-time $M \times [0, t]$ with a Lorentzian metric. The evolution operation $Z_t : \mathcal{H}(M) \to \mathcal{H}(M)$ is described via a Feynman integral. The operators Z_t satisfy *composition law*:

$$Z_s \circ Z_t = Z_{s+t}.$$

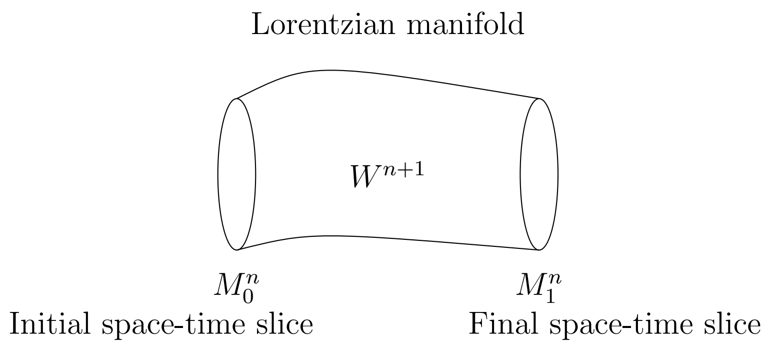


FIGURE 1. Time evolution of M_0^n to M_1^n

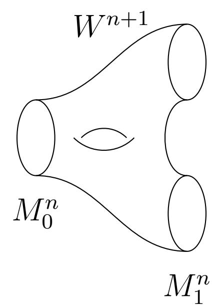


FIGURE 2. Time evolution of M_0^n to M_1^n

More generally, W^{n+1} need not have a product structure. Even in the presence of non-trivial topology of W^{n+1} and , the framework remains valid (Figure 2). To get a quantum field theory, we assume that the quantum field theory depends only on topology and not on metric. Additionally, we assume the following hold:

(1) Composition law: if W^{n+1} is the evolution of M_0^n to M_1^n and W'^{n+1} is the evolution of M_1^n to M_2^n then W^{n+1} glued with W'^{n+1} is the evolution of M_0^n to M_2^n, see Figure 3.
(2) Locality: if $M^n = N_0^n \sqcup N_1^n$ is a disjoint union, then
$$\mathcal{H}(M^n) \cong \mathcal{H}(N_0^n) \otimes \mathcal{H}(N_1^n),$$
expressing the independence of two disjoint systems in quantum field theory.

1.1.1. *Background definitions.* We will record some definitions of category theory to define topological quantum field theory. For a background on category theory see [ML98].

Definition 1.2. A *category,* $\mathcal{C}$, consists of:

- A class of *objects* $(A, B, C \dots)$.
- For each pair of objects A, B, a set of *morphisms* $\mathbf{Hom}_{\mathcal{C}}(A, B)$.
- *An associative composition operation:* $\mathbf{Hom}_{\mathcal{C}}(A, B) \times \mathbf{Hom}_{\mathcal{C}}(B, C) \to \mathbf{Hom}_{\mathcal{C}}(A, C)$.
- For each object, an *identity element* $\mathrm{id}_A \in \mathbf{Hom}_{\mathcal{C}}(A, A)$ which composes trivially.

Example 1.3. (1) $\mathbf{Vect}_{\mathbb{C}}$, **Groups**, **Rings**, **Set**, **Top** etc.

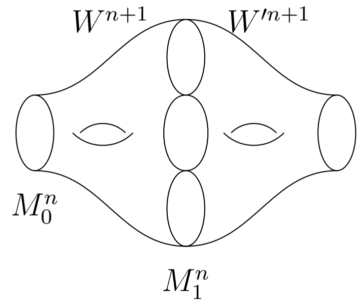


FIGURE 3. Composition of two space-times

(2) Given a graph G, the *path category* of G consists of vertices as objects and $\mathbf{Hom}(v, w) = \{\text{set of all edge paths } v \to w\}$.
(3) A group G is a category with one object where every morphism is invertible.

Definition 1.4. The $n+1$-*dimensional cobordism category,* $\mathbf{Cob}_{n+1}$, consists of the following data:

- Objects consists of closed oriented n-dimensional smooth manifolds.
- A morphism from $M_0^n \to M_1^n$ is an $n+1$-manifold W^{n+1} with boundary ∂W divided into two parts (in and out) equipped with orientation preserving diffeomorphisms to $\overline{M}_0$[1] and M_1. We consider the morphisms up to orientation preserving diffeomorphism relative to boundary, i.e., ones which restrict compatibly to the boundary identification. See Figure 4.

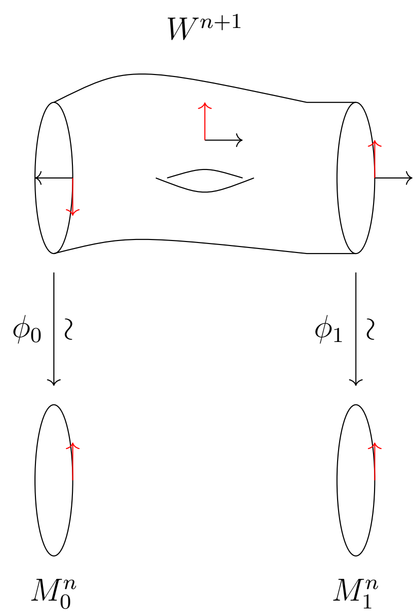


FIGURE 4. Identity in $\mathbf{Cob}_{n+1}$

[1] $\overline{M}$ is M with opposite orientation.

- We can compose $W : M_0 \to M_1$ with boundary identifying map ϕ_1 and $W' : M_1 \to M_2$ with boundary identifying map ψ_1 by gluing W and W' via the identification $\psi_1^{-1}\phi_1$ to get $W \circ_{\psi_1^{-1}\phi_1} W'$, see Figure 3.
- The identity element is $M \times I : M \times \{0\} \to M \times \{1\}$. The fact that the identity composes trivially is a consequence of the collar neighborhood theorem (Figure 5).

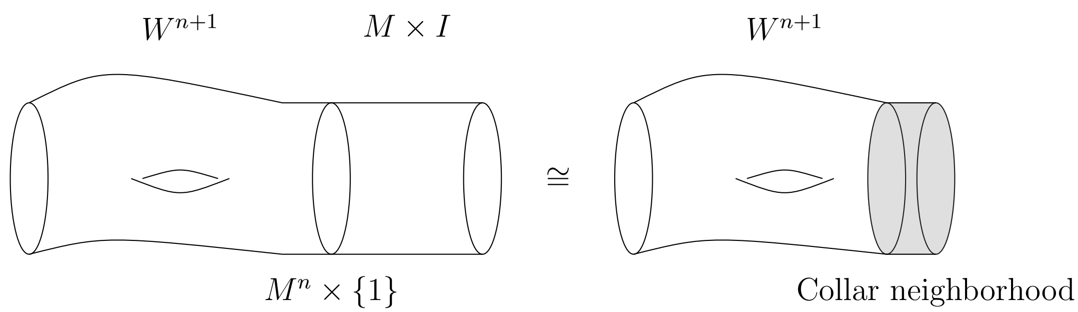


FIGURE 5. Identity in $\mathbf{Cob}_{n+1}$

Remark 1.5. $M \times I$ can be regarded as a cobordism between $\overline{M} \sqcup M$ and $\emptyset$ (Figure 6).

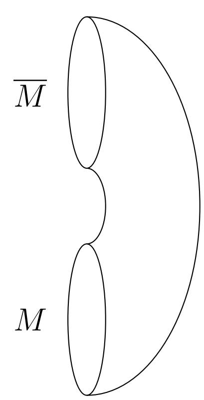


FIGURE 6. Cobordism of $\overline{M} \sqcup M$ and $\emptyset$

Definition 1.6. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to $\mathcal{D}$ is a rule which maps objects and morphisms

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "f"] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, mapsto] & F(B) \end{tikzcd}$$

preserving composition and identity.

Definition 1.7. A category $\mathcal{C}$ is *symmetric monoidal* if the following hold:

- Monoidal structure: There is an associative and unital bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, i.e., given two objects A, B, there exists an object $A \otimes B$ and for two morphisms $A \to C$ and $B \to D$ there is a corresponding morphism $A \otimes B \to C \otimes D$. We impose
$$A \otimes (B \otimes C) \overset{\alpha_{A,B,C}}{\cong} (A \otimes B) \otimes C$$
Further, there exists a unit object $\mathbb{1} \in \mathcal{C}$ sch that $\mathbb{1} \otimes A \cong A \cong A \otimes \mathbb{1}$.
- Symmetry: Given two objects A, B, we have
$$A \otimes B \overset{\sigma_{A,B}}{\cong} B \otimes A$$
such that the composition $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \xrightarrow{\sigma_{B,A}} A \otimes B$ is the identity.

We say that $(\mathcal{C}, \otimes)$ is a *symmetric monoidal category.*

Exercise 1. (1) $(\mathbf{Cob}_{n+1}, \sqcup)$ is a symmetric monoidal category, where $\sqcup$ is the disjoint union. Check that the idenity element is $\emptyset$.
(2) $(\mathbf{Vect}_{\mathbb{C}}, \otimes_{\mathbb{C}})$ is a symmetric monoidal category, where $\otimes_{\mathbb{C}}$ is the tensor product of complex vector spaces. Check that the identity element is $\mathbb{C}$.

Remark 1.8. We impose that a functor $F : (\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$ between two symmetric monoidal categories preserves $\otimes$ structure. Namely,
$$F(A \otimes B) \cong F(A) \otimes F(B), \quad F(\mathbb{1}) \cong \mathbb{1}.$$

2. Definition of TQFT

Definition 2.1. An $n+1$-*dimensional TQFT* is a tensor-functor
$$(\mathbf{Cob}_{n+1}, \sqcup) \to (\mathbf{Vect}_{\mathbb{C}}, \otimes_{\mathbb{C}}). \tag{2.1}$$

Example 2.2. Suppose Z is a TQFT and W^{n+1} is a closed $n+1$ manifold divided into half W_0 and W_1 by an n-manifold C. Then we can view W^{n+1} as a composition of morphsims
$$\emptyset \xrightarrow{W_0} C \xrightarrow{W_1} \emptyset.$$
See Figure 7.

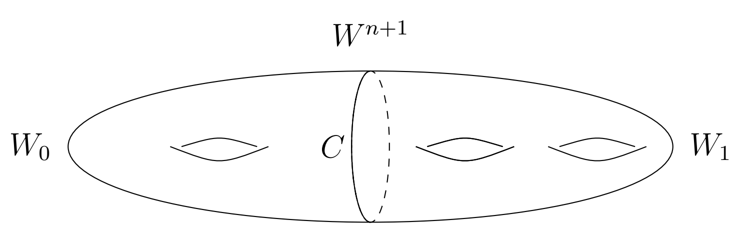


Figure 7. Composition of cobordisms between empty set and C and C and empty set

Applying Z we get
$$Z(\emptyset) = \mathbb{C} \xrightarrow{Z(W_0)} Z(C) \xrightarrow{Z(W_1)} \mathbb{C}.$$

Since $Z(W_0)$ is a linear map $\mathbb{C} \to Z(C)$ we can view $Z(W_0)$ as an element of $Z(C)$. Further, since $Z(W_1) \in \mathbf{Hom}_{\mathbf{Vect}}(Z(C), \mathbb{C})$ and the composition map is in $\mathbf{Hom}_{\mathbf{Vect}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, we can view

$$Z(W) = \langle w_0 | w_1 \rangle \in \mathbb{C},$$

where $\langle w_0 | \in Z(C)$ and $| w_1 \rangle \in Z(C)^*$. Namely, $Z(W)$ is a number.

Remark 2.3. (1) In general, we will prove that $Z(W)$ is a topological (numerical) invariant of W.
(2) The vector spaces associated to n-manifolds are also topological invariants. However, in most applications, we prefer to work with numerical invariants.

3. Example: 0+1 dimensional TQFT

There are two oriented connected 0-manifolds. A point with positive orientation $\textcolor{red}{\bullet}$ and a point with negative orientation $\textcolor{blue}{\bullet}$. Therefore, the objects of $\mathbf{Cob}_{0+1}$ are just finite sets of oriented points. Given a TQFT, we have $Z(\textcolor{red}{\bullet}) = V$ and $Z(\textcolor{blue}{\bullet}) = W$ for some vector spaces V and W. In general, we can use the tensor product axiom to evaluate Z on a collection of oriented points. Some examples of morphism are given below.

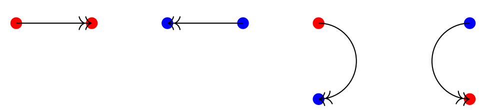


Figure 8. Bordisms of 0-smooth manifolds

(1) The first one in Figure 8 gets mapped to id_V under Z.
(2) The second one in Figure 8 gets mapped to id_W under Z.
(3) The third one in Figure 8 gets mapped to $V \otimes W \xrightarrow{\beta} \mathbb{C}$. Physically, this corresponds to two dots annihilating.
(4) The fourth one in Figure 8 gets mapped to $\mathbb{C} \to W \otimes V$, which by linearity can be viewed as an element b of $W \otimes V$. Physically, this corresponds to two dots forming.

We can compose all of the four bordism as in Figure 9 and noting that bordisms are defined relative to the boundary we get equivalence of the bordisms as in Figure 9.

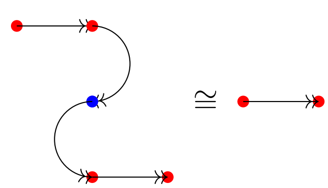


Figure 9. Composing bordisms relative boundary

The equivalence of bordisms in Figure 9 implies that after applying Z we get that the composition of $V \to V \otimes W \otimes V \to V$ is id_V. Formally, it means that

$$(\beta \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes b) = \mathrm{id}_V .$$

As a consequence of the above algebraic relation, we can prove that V is finite dimensional. In fact, since $b \in W \otimes V$ we can write a finite sum

$$b = \sum w_i \otimes v_i.$$

Then the composition map $V \to W \otimes V \otimes V \to V$ is

$$v \mapsto \sum \beta(v, w_i)v_i.$$

Since the sum is finite, the image is a finite dimensional subspace of V. Since the composition is identity, it means that V is finite dimensional.

A further consequence is that β is a perfect (i.e., non-degenerate) pairing $V \hookrightarrow W^*$. Using a similar argument, we can get a perfect pairing $W \hookrightarrow V^*$.

Exercise 2. Under the identification $W \cong V^*$ using β, we can write $b \in V \otimes V^*$ as

$$b = \sum e_i \otimes e_i^*$$

where e_i forms a basis of V and e_i^* forms a basis of V^* such that $e_i^*(e_j) = \delta_{ij}$. Furtherfore, under this identification, β is the map $V \otimes V^* \ni v \otimes \phi \mapsto \phi(v)$.

Similarly, we can compose the fourth one and the third one in Figure 9 after permuting the boundary points in third one to get cobordism between $\emptyset$ with itself, see Figure 10. After

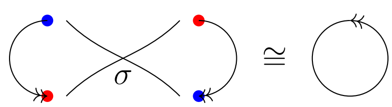


FIGURE 10. Composing bordisms after permutation

applying Z and using Exercise 2, we get a map $\mathbb{C} \to V \otimes V^* \to V^* \otimes V \to \mathbb{C}$ where the first map is an element b and the third map is β such that under the composition, we get

$$\sum e_i \otimes e_i^* \overset{\sigma}{\mapsto} \sum e_i^* \otimes e_i \mapsto \sum e_i^*(e_i) = \dim V.$$

Therefore, Z of a circle recovers the dimension of V, a natural number. Compare this computation with Example 2.2.

Remark 3.1. In our definition of a TQFT, the dimension of the vector spaces we allowed to be infinite. However, we saw that the $0+1$ dimensional TQFT assigns finite dimensional vector spaces. In particular, asking a quantum field theory to be functorial imposes a restriction that the state space for 0 objects to be a finite dimensional vector space. The argument works in $n+1$ TQFT as well. Namely, if $Z(M^n) = V$ and $Z(\overline{M}^n) = W$ then we can use equivalence of cobordisms as in Figure 11 to get

$$W \cong V^*$$

and that both are finite dimensional such that

$$Z(M \times S^1) = \dim V \in \mathbb{N} \subset \mathbb{C}.$$

This imposes that topological invariants of $M \times S^1$ that are not natural numbers can't be topological TQFT. In particular, most invariants of 3 manifolds are not coming from TQFT

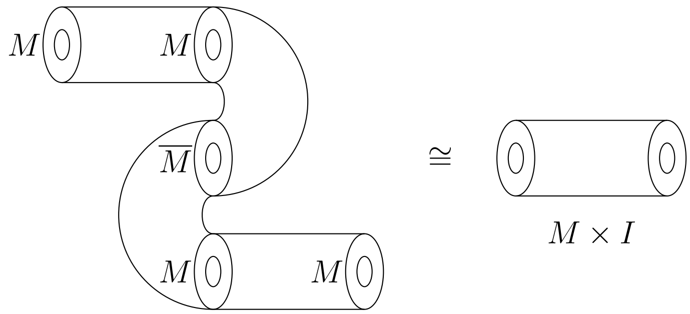


FIGURE 11. Duality of $Z(M^n) = V$ and $Z(\overline{M}^n) = W$

In general, there is a deep fact that a TQFT corresponds to *fully dualizable objects*, a concept that incorporates boundedness condition like finite dimensionality and compactness. To get situations where we get infinite dimensional Hilbert spaces (which is the most interesting situation to physicists), we have to tweak certain axioms on what a TQFT means.

Finite dimensionality basically followed by being able to join two bordisms to get $M \times S^1$. As long as we are allowed to close off cups with caps to make $M \times S^1$, a similar diagramatic proof shows that the vector spaces assigned are finite dimensional. Not allowing the existence of two sided caps is a way to tweak the axiom.

Adding a metric forces not to always have an identity. So we lose monoidal structure. This might lead to infinite dimensional vector spaces.

4. Representation of mapping class groups

Definition 4.1. The *mapping class group* MCG of a smooth closed manifold M^n is defined as

$$\mathrm{MCG}(M^n) \coloneqq \pi_0 \,\mathrm{Diff}^+(M),$$

the path components of the group of orientation preserving self-diffeomorphism of M, or equivalently the isotopy classes of diffeomorphisms.

Example 4.2.
(1) $\mathrm{MCG}(\mathbb{T}^2) = \mathrm{SL}(2,\mathbb{Z})$, given by the quotient of the natural action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$. For instance, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shears the unit square with vertices $(0,0), (0,1), (1,0)$ and $(1,1)$.
(2) $\mathrm{MCG}(S^1)$ is trivial because $\mathrm{Diff}^+(S^1)$ is homotopy equivalent to $\mathrm{SO}(2)$, which has one path component.
(3) $\mathrm{MCG}(\Sigma_{g\geq 2})$ turn out to be interesting discrete groups.

Given an $n+1$ dimensional TQFT, Z, the $\mathrm{MCG}(M^n)$ acts on the space $Z(M^n)$, giving a representation of $\mathrm{MCG}(M^n)$. More precisely, each $\phi \in \mathrm{Diff}^+(M^n)$ defines a mapping cylinder (Figure 12). Recall that we defined bordisms up to diffeomorphism relative to the boundary. Therefore, the mapping cylinder is a bordism from M to itself. Note that isotopic ϕ give rise to diffeomorphic cylinder. In any case, such a bordism acts on $Z(M)$ by "$Z(\phi)$." Further, if ψ defines another mapping cylinder then $Z(\phi \circ \psi) = Z(\phi) \circ Z(\psi)$ holds because of the functoriality of Z. Therefore, we get a representation of $\mathrm{MCG}(M^n)$ on $Z(M)$.

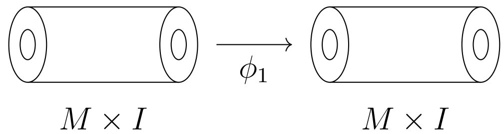


FIGURE 12. Mapping cylinder of M where the middle is glued using the map $\phi \in \text{Diff}^+$

In any case, people who are interested in understanding the mapping class group study their representation arising from TQFTs.

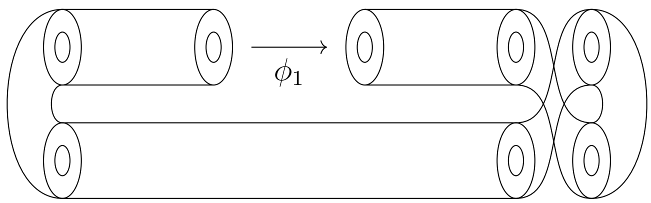


FIGURE 13. Mapping torus of M where the middle is glued using the map $\phi \in \text{Diff}^+$

We can compose the mapping cylinder with three $M \times I$ to get the mapping torus T_ϕ of ϕ (Figure 13) and use functoriality of Z to compute $Z(T_\phi)$ by noting that

$$\sum e_i \otimes e_i^* \mapsto \sum_i (Z(\phi)e_i) \otimes e_i^* \mapsto e_i^*(Z(\phi)e_i) = \langle e_i | Z(\phi)e_i \rangle = \operatorname{tr} Z(\phi).$$

In particular,

$$Z(T_\phi) = \operatorname{tr} Z(\phi),$$

the character of the representation of ϕ. The above formula is a generalization of the fact that $Z(M \times S^1) = \dim Z(M)$.

5. 1+1 DIMENSIONAL TQFT

Recall that every 1-manifold is diffeomorphic to a finite disjoint union of S^1. Therefore, a $1+1$ TQFT is determined once we fix $Z(S^1) = V$ and $Z(\overline{S}^1) = W$. As discussed before, we know that $W \cong V^*$ and $V = W^*$ are finite dimensional vector spaces. In this section, we will prove that V has more algebraic structure using the existence of certain bordisms in $\mathbf{Cob}_{1+1}$ and their topological properties, namely, same shapes can be built of of different pieces.

First, since there exists an orientation preserving diffeomorphism $S^1 \to \overline{S}^1$ via a flip (reflection) we know that $V = W$. Therefore, after identifying every circle with the standard S^1 via such a flip if necessary we can consider the standard pairing, β, corresponding to the bordism from $S^1 \sqcup S^1$ to $\emptyset$ (Figure 14) as a pairing from $V \otimes V \to \mathbb{C}$ instead of $V \otimes W \to \mathbb{C}$.

A pair of pants (Figure 15 which can be thought of two circles coalescing to form one) corresponds under Z to a map

$$\mu : V \otimes V \to V.$$

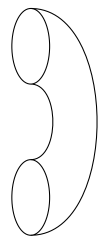


FIGURE 14. Cobordism of $S^1 \sqcup S^1$ to $\emptyset$

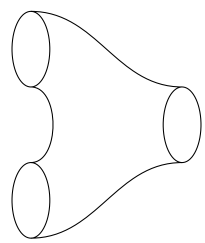


FIGURE 15. Pair of pants

A disc with outgoing circle (cup) (Figure 16) corresponds to a map $\mathbb{C} \to V$ which can be regarded as an element in V. We can call it to be 1.

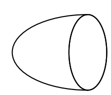


FIGURE 16. Unit

Composing a cup and a cylinder with a pair of pants, we still get a cylinder, see Figure 17. At an algebraic level, we have

$$\mu(v, 1) = v = \mu(1, v). \tag{5.1}$$

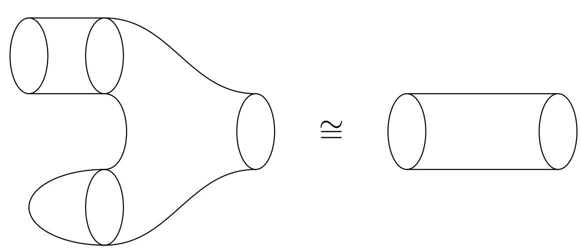


FIGURE 17. Cap and cylinder composed with pair of pants

Observe that the pair of pants is diffeomorphic to pair of pants after flipping legs (Figure 18). At an algebraic level, we get

$$\mu(v, w) = \mu(w, v).$$

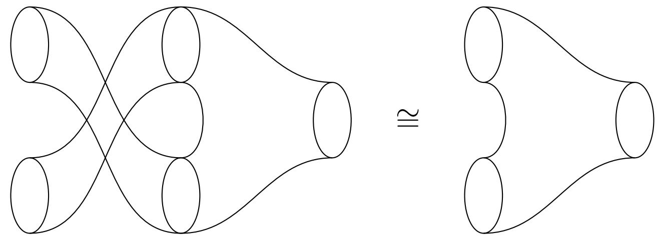


FIGURE 18. Symmetry

It tursns out that

$$\langle \omega \rangle_0 = \sum_{n=0}^{r-2} \Delta_n^2 =$$

We also get associativity of μ by looking at Figure 19. Namely,

$$\mu(\mu(u,v),w) = \mu(u,\mu(v,w)).$$

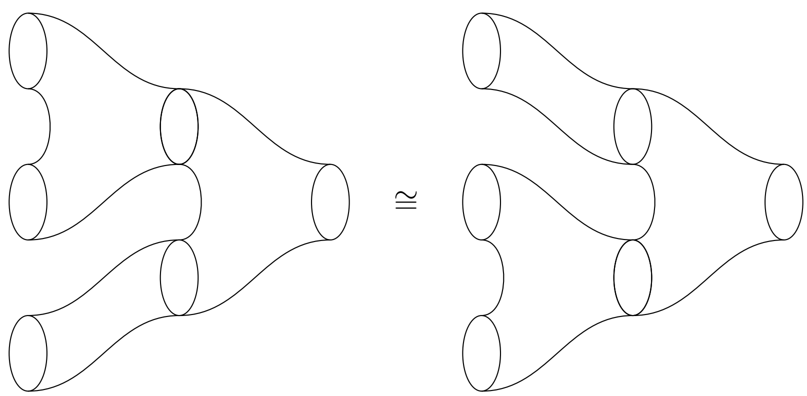


FIGURE 19. Associativity

We have proven the following proposition.

Proposition 5.1. *V is a unital, associative commutative algebra.*

Henceforth, we will rename $V = A$.

On the other hand, a backward pair of pants (Figure 20) gives us a coassociative, counital, cocommutative, coproduct

$$\Delta : A \to A \otimes A$$

Here, co-associativitiy means

$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta.$$

It follows from using a diagram similar to Figure 19.

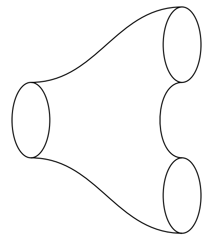


FIGURE 20. Co-product

Counital means that corresponding to a disc with ingoing circle, a bordism $S^1 \to \emptyset$ (Figure 21) there is a co-unital structure

$$\epsilon : A \to \mathbb{C}$$

satisfying

$$(\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A \,.$$

The above algebraic relation corresponds to Figure 22.

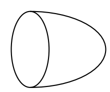


FIGURE 21. Co-unit

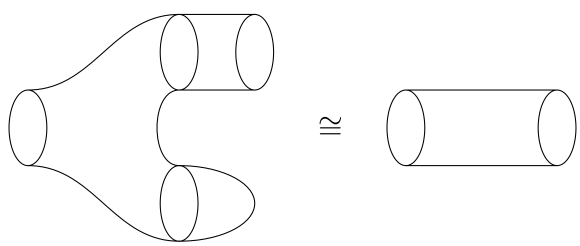


FIGURE 22. Co-unital structure

In particular, we have proven the following proposition.

Proposition 5.2. *A is a counital, coassociative, cocommutative coalgebra.*

Remark 5.3. A is not a Hopf algebra.

However, Δ and μ satisfy a ''Frobenius relation." Namely, corresponding to the composition of pair of pants facing in opposite direction (Figure 23), we get an algebraic relation

$$\Delta \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta).$$

Remark 5.4. What we saw is that $1 + 1$TQFT is highly overdetermined as a consequence of that fact that the same surface can be formed out of different bits giving rise to redundancy.

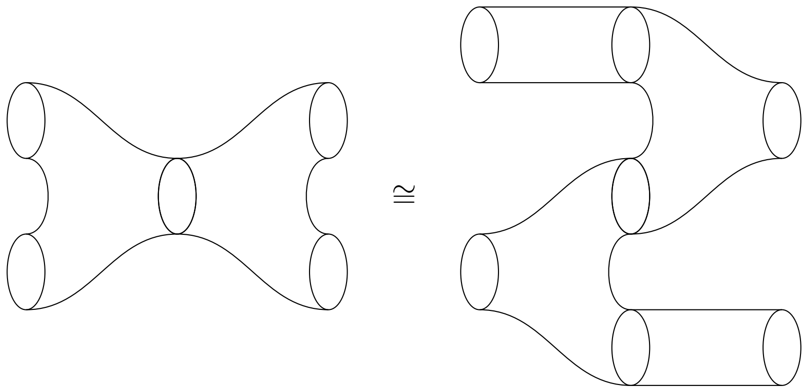


FIGURE 23. Frobenius relation

As an example of redundancy, observe that there exists a non-degenerate pairing on A

$$\begin{aligned} A \otimes A &\mapsto \mathbb{C} \\ a \otimes b &\mapsto \epsilon(\mu(a,b)) \end{aligned}$$

corresponding to capping off a pair of pant as in Figure 24. The non-degeneracy of the pairing follows because of the algebraic relation corresponding to Figure 25.

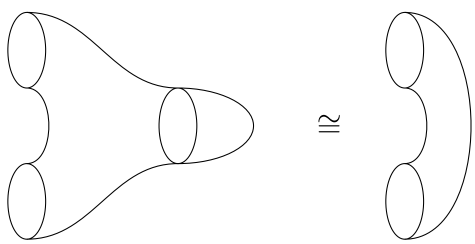


FIGURE 24. Non-degenerate pairing

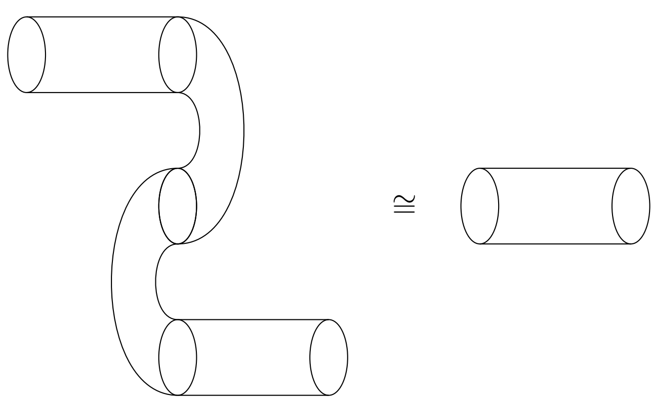


FIGURE 25. Non-degeneracy

Actually, this non-degenerate pairing and μ determine Δ completely. In fact, corresponding to Figure 26.

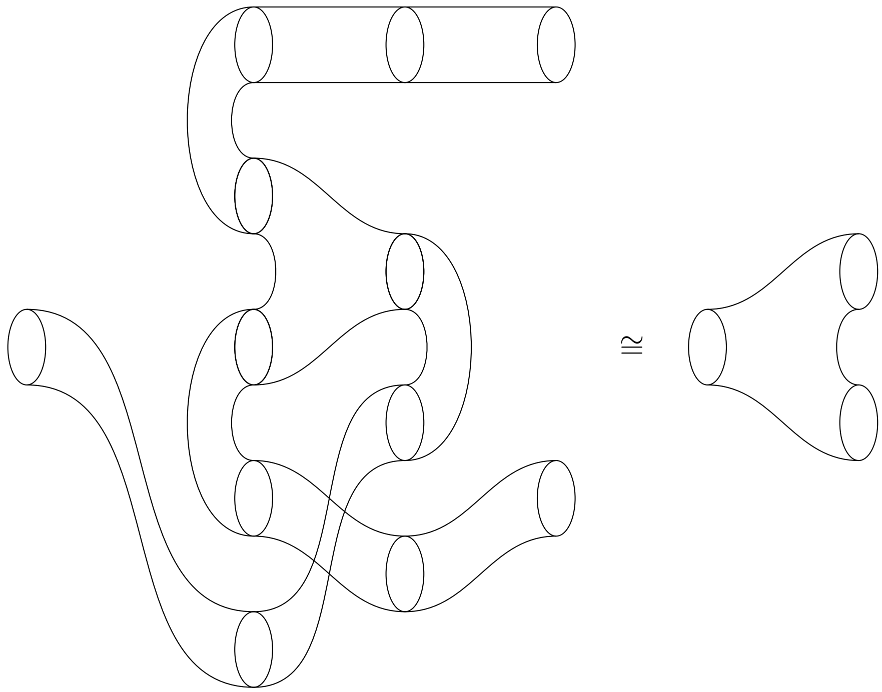


FIGURE 26. Building coproduct from the non-degenerate pairing and product μ

Definition 5.5. A *Frobenius algebra* is a finite dimensional associative unital algebra with a product μ and a counit ϵ defining a non-degenerate pairing: $a \otimes b \mapsto \epsilon(\mu(a, b))$.

Remark 5.6. Once we have non-degenerate pairing, we can generate a co-unit. So, we can instead define a Frobenius algebra using co-product and product satisfying the Frobenius relation. However, our definition is fairly minimal and avoids using coproduct which can be hard to understand as a map that sends $a \in A$ to $\sum a_i \otimes b_i \in A \otimes A$.

Theorem 5.7. *A $1+1$ dimensional TQFT defines a commutative Frobenius algebra. Conversely, a commutative Frobenius algebra determines a $1+1$-dimensional TQFT.*

We will postpone the sketch of the proof until §6.

So far, we have seen that $1 + 1$ dimensional TQFT deterimes a commutative Frobenius algebara structure on $Z(S^1)$ with product μ and co-unit ϵ and a non-degenerate pairing $\beta = \epsilon \circ \mu$ corresponding to simple surfaces in Figure 15, Figure 21 and Figure 24 respectively. The coproduct is Figure 20, unit is Figure 16 and copairing is 28.

To prove the theorem, we must understand the generators and relations of $\mathbf{Cob}_{1+1}$. The idea is to cut any cobordism into these elementary pieces (generators) and write them into words and reduce the word using corresponding algebraic relations. However, we want the different way of cutting a surface into pieces to give the same answer. It turns out that different ways of cutting bordisms translate to the expressions in the structure constants of the Frobenius algebra which are equal modulo the relations that define the Frobenius algebra.

The strategy of the proof of the converse is as follows:

(1) Set $Z(\sqcup_i S^1) = \otimes_i A$ and use the Frobenius algebra structure to define the values of Z(simplest surfaces).
(2) For more general surfaces as cobordisms from some disjoint copies of S^1 to other disjoint copies fo S^1, we need to associate appropriate linear maps $\otimes_i A \to \otimes_j A$. To do so, we need to do the following.
  (a) Every surface cobordism can be factored into composite of "simplest surface."
  (b) We need to show that all relations among such (different) decomposition of surfaces come from the Frobenius algebra relations. For instance, we know that associativity in the Frobenius algebra implies $\mu \circ (\mathbb{1} \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1})$ in the Frobenius algebra. We will see that these relation transfer to the equivalence of the surfaces in Figure 19. The proof uses Morse theory.

5.1. **Examples of 1+1 TQFT.** Before giving the proof of Theorem 5.7, we list some explicit examples of $1+1$ TQFT.

Example 5.8. Consider a semi-simple commutative algebra $\bigoplus \mathbb{C} e_i$ (which can be thought of as an algebra of diagonal matrices) such that $e_i^2 = e_i$. The identity element is $\mathbb{1} = \sum e_i$. Further, the counit ϵ satisfies $\epsilon(e_i) = \lambda_i \in \mathbb{C}$ for some arbitrary λ_i. Note that e_i gets paired to $\lambda_i e_i^*$ under $A \cong A^*$.

Recall that $Z(S^1 \times S^1) = n = \dim A$.

The unital and co-unital structure enforce that $S^1 \sqcup S^1 \to \emptyset$ (Figure 27) gets mapped to $\sum \lambda_i e_i^* \otimes e_i^*$.

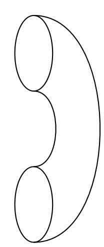


FIGURE 27. $S^1 \sqcup S^1 \to \emptyset$

Similarly, $\emptyset \to S^1 \sqcup S^1$ (Figure 28) gets mapped to $\sum \lambda_i^{-1} e_i \otimes e_i$.

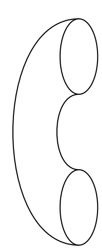


FIGURE 28. $\emptyset \to S^1 \sqcup S^1$

Further, composing $\emptyset \to S^1 \sqcup S^1$ (Figure 28) with $S^1 \sqcup S^1 \to S^1$, the multiplicative structure, we get a one-holed torus. Then the one-hold torus gets mapped to $\sum \lambda_i^{-1} e_i$ (Figure 29).

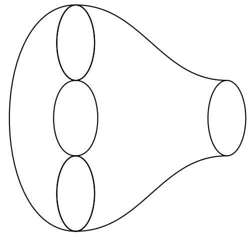


FIGURE 29. One holed torus

Finally, composing g one holded torus with multiplication multiple times and capping off the rightmost hole (see Figure 30), we get a surface Σ_g. Therefore,

$$Z(\Sigma_g) = \epsilon(\sum \lambda_i^{-g} e_i) = \sum \lambda_i^{1-g} \in \mathbb{C}.$$

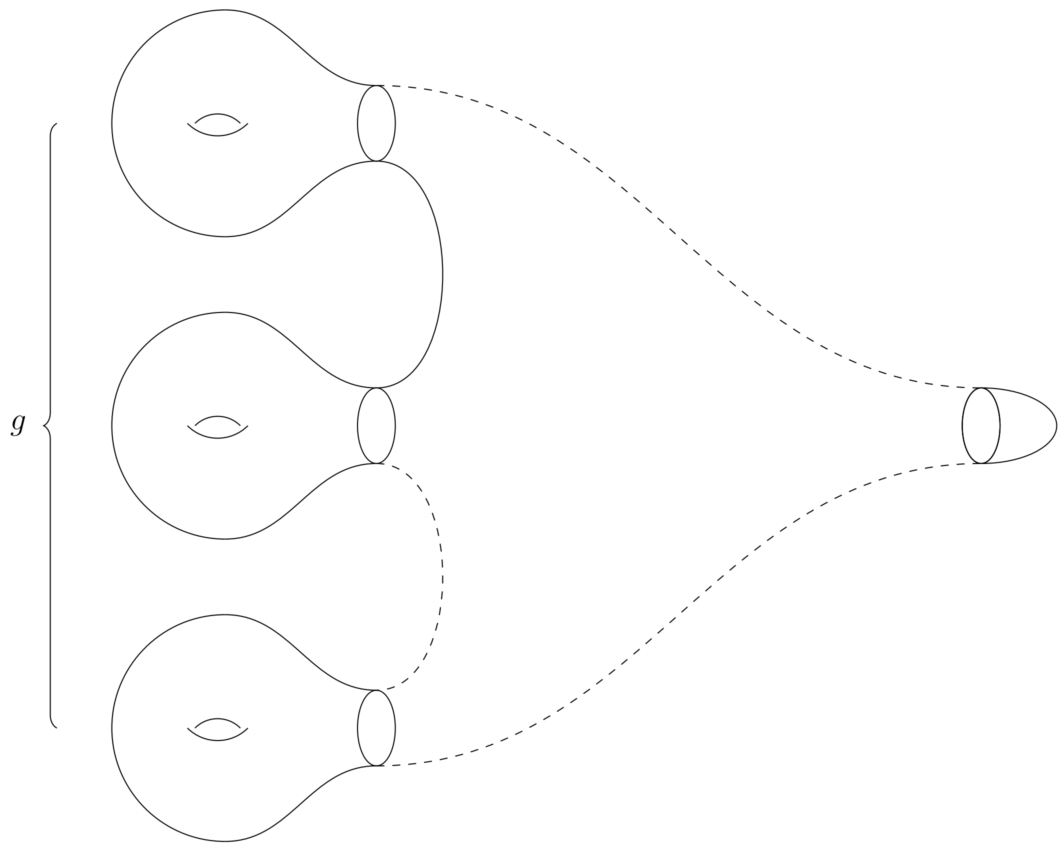


FIGURE 30. A surface Σ_g with genus g

Example 5.9. Consider a non-semisimple algebra $A \coloneqq \mathbb{C} \cdot \mathbb{1} \oplus \mathbb{C} \cdot X$ where $X^2 = 0$ and $\epsilon(1) = 1$, $\epsilon(X) = 1$. Under the previous restriction, we know that the non-degenerate pairing sends

$$1 \otimes 1 \mapsto 0$$

$$1 \otimes X \mapsto 1$$
$$X \otimes 1 \mapsto 1$$
$$X \otimes X \mapsto 0.$$

Further, the copairing is given by

$$1 \otimes X + X \otimes 1 \in A \otimes A. \tag{5.2}$$

The pairing and co-pairing together with the decomposition a torus with a hole as in Figure 29 imply that a one-holed torus corresponds to $2X \in A$. Similarly, using the decomposition of Σ_g as in Figure 30, we know that

$$Z(\Sigma_g) = \epsilon((2X)^g) = \begin{cases} 0 & g \geq 2, \\ 2 & g = 1, \\ 0 & g = 0. \end{cases}$$

Note that $2 = Z(S^1 \times S^1) = \dim A$.

Remark 5.10. The algebra in Example 5.9 is called “Khovanov's Frobenius algebra.” It appears in Knovanov's homology (of knots), which is a categorification of the Jones polynomials in the sense that they are graded homology groups so that the dimension is a polynomial. And the alternating sum of polynomial is the graded dimension giving rise to Jones polynomial.

Example 5.11. As general version of Khovanov's Frobenius algebra, we can take $A = H^*(Y)$, where Y is closed oriented manifold of dimension d with $\epsilon(\bullet) = \langle \bullet, [Y] \rangle$ where $[Y] \in H_d(Y, Z) = \int_Y \bullet$ to define a TQFT. Namely, In Example 5.9, $A = (H^*(S^2), \int_{S^2})$. The non-degeneracy of the (cup) product follows from the Poincaré duality.

Remark 5.12. The claim in 5.11 is not literally true because $H^*(Y)$ is not a commutative ring but a graded-commutative ring in the sense that

$$a \cup b = (-1)^{\deg a \cdot \deg b} b \cup a$$

for homogeneous elements a and b. Nevertheless, it is a commutative Frobenius algebra object in the category of $\mathbb{Z}$-graded vector spaces. The target category of a TQFT is Gr **Vect**, where the symmetric monoidal structure satisfies the symmetry rule

$$V \otimes W \cong W \otimes V$$
$$v \otimes w \mapsto (-1)^{\deg v \deg w} w \otimes v$$

In general, we can extend our notion of TQFT to be a functor from $\mathbf{Cob}_{n+1}$ to any symmetric monoidal category. Then a general version of Theorem 5.7 is

> $\mathbf{Cob}_{1+1}$ is a free symmetric monoidal category on a commutative Frobenius algebra object.

More precisely,

$$\otimes \mathbf{Fun}(\mathbf{Cob}_{n+1}, \mathcal{C}) = \text{CFA objects in } \mathcal{C},$$

where $\otimes \mathbf{Fun}(\mathbf{Cob}_{1+1}, \mathcal{C})$ is the category of tensor functors from $\mathbf{Cob}_{1+1}$ to a symmetric monoidal cateogry $\mathcal{C} is$ and CFA means commutative Frobenius algebra object. This is a statement of freeness. For comparison, a homomorphism from a free group F to any other group G is an assignment of an element in G. The above statement is at a functorial categorical level. It says that $\mathbf{Cob}_{1+1}$ is generated by tensor powers of S^1. The basic structural elements (co-unit, unit, product, co-product) satisfy relations of a Frobenius algebra.

6. A crash course on Morse theory

To prove Theorem 5.7, we hand-wave some facts from Morse theory.[2] The main idea of Morse theory is to understand topological properties of a smooth manifold using the critical points of a smooth function (say for instance height function). More precisely, using the critical points of a generic *Morse* function, we will get a (handle-body decomposition of a surface (and more generally a manifold). Then we will sketch that varying Morse functions changes the decomposition in a way corresponding to defining relations in a Frobenius algebra. For more details see [Koc04].

Definition 6.1. Let M^n be a smooth manifold and $f : M^n \to \mathbb{R}$ be a smooth function. A *critical point* $p \in M$ of f is where

$$df_p = 0.$$

Here, $df_p : T_pM :\to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is the differential of f at p.

Example 6.2. The height function $h : \mathbb{T}^2 \to \mathbb{R}$ on the torus that gives height of each point is smooth and has critical four critical points as shown in Figure 31.

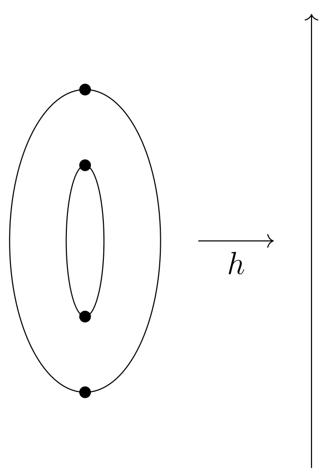


Figure 31. Height function on the torus has critical points on •.

Remark 6.3. For dimension counting reasons we expect that critical points are isolated points. In fact, $df_p = 0$ is a codimension n condition. Therefore, such points p should be "isolated." The isolation is not always true for height function if we have flat pieces on a manifold. Nevertheless, the isolation of critical points is generalically true.

[2]Morse theory is probably the most important tool in the manifold theory. The history of Morse theory dates back at least to Maxwell's paper [Max70] and later to [Mor28]. A detail account of Morse theory is in [Mil63].

Associated a critical point p is a quadratic form called the *Hessian* defined as

$$\begin{aligned} T_pM \times T_pM &\to \mathbb{R} \\ (X, Y) &\mapsto (\tilde{X}\tilde{Y}f)_p, \end{aligned}$$

where $\tilde{X}$ and $\tilde{Y}$ are extension of X and Y at a point. The Hessian is symmetric because $[\tilde{X}, \tilde{Y}]f = 0$ since $df = 0$. In local coordinates, it is the matrix of second partial derivatives. Equivalently, it is the second order part of the Taylor series of f at p. Generically, we expect the Hessian to be non-degenerate and therefore we hop to diagonalize such that after choosing a basis it has ± 1 on its diagonal.

Definition 6.4. We say that p is a non-degenerate point if the Hessian is non-degenerate. In that case, if the Hessian has k negative eigenvalues and $n-k$ positive eigenvalues, we say that p is an $\operatorname{ind} k$ (index k) non-degenerate critical point.

Lemma 6.5 (Morse)**.** *If p is a non-degenerate* $\operatorname{ind} k$ *critical point then there exists a coordinate system $(x_1, \ldots, x_n)$ near p such that locally*

$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{k+1}^{n} x_i^2. \tag{6.1}$$

Remark 6.6. The expression on the right of (6.1) is called the normal form. When $n = 2$ case, a function in its normal form near a non-degenerate critical point looks either like an upsided parabola, a saddle and or a down-sided parabola, see Figure 32.

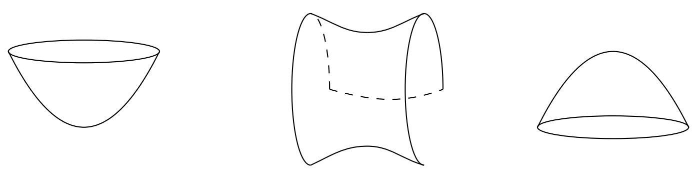


FIGURE 32. Heuristic profile of surfaces defined by functions of normal form in $n = 2$, $k = 0, 1, 2$ respectively.

Definition 6.7 (Pre definition)**.** A Morse function on M is a smooth function $f : M \to \mathbb{R}$ with non-degenerate isolated critical points.

Theorem 6.8. *Morse functions are generic, i.e., they form an open dense set in the space of all smooth functions.*

6.1. **Gradient flow lines.** Fix a Riemannian metric g on M. Then we can associate to $df_p \in T_pM^*$ a vector field $\nabla f_p \in T_pM$ via the pairing

$$g(\nabla f_p, \bullet) = df_p(\bullet).$$

The gradient points in the "steepest descent" direction.

Remark 6.9. Note that df_p is a one form. It can be thought of as a folition of T_pM by the parallel of $\ker df_p$. The parallel copies can be thought of as infinitesimal contour lines in the tangent space. A metric g allows us to define orthogonal direction (depending on the metric) to $\ker df_p$. These level sets give the direction of steepest descent of the vector field ∇f.

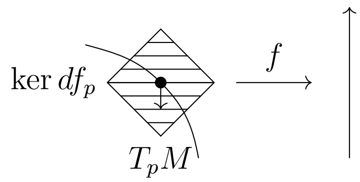


FIGURE 33. Direction of steepest descent

The *gradient flow* lines $\gamma(t)$ is the flow lines of the vector field ∇f :

$$\frac{d}{dt}\gamma(t) = -\nabla f_{\gamma(t)}.$$

Proposition 6.10. *Suppose that M is compact. Then the gradient flow lines of a Morse function f will converge as $t \to \pm\infty$ to critical points of f.*

Example 6.11. The gradient flow lines on a symmetric torus (of the height function) is depicted in Figure 34 where $\bullet$ represent critial points. Intuitively, we can think of the Proposition 6.10 as rain drops starting from top critical point end up at the critical points below.

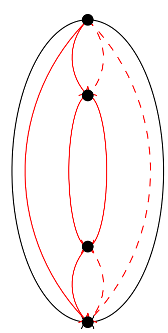


FIGURE 34. Gradient-flow of a Morse function on a symmetric torus

However, the height function on a symmetric torus does not give rise to generic gradient flow lines. In fact, if we tilt the torus a little bit then there will be no flow lines that start and end at the critical points with index 1 (Figure 35) unlike in the symmetric setting (Figure 34). See Assumption 6.14 for more details.

Every critical point consists of a *descending manifold* and an *ascending manifold*

$$D_p \coloneqq \{\cup \text{ all flow lines emerging from } p\}$$
$$A_p \coloneqq \{\cup \text{ all flow lines converging to } p\}.$$

Here, γ emerges from p means that $\lim_{t\to-\infty}\gamma(t) = p$ and it converges to p means that $\lim_{t\to\infty}\gamma(t) = p$.

Example 6.12. Consider an index 1 critical point in dimension 2 so that the surface is defined by $f = x^2 - y^2$. In this case, D_p is the curve $f = x^2$ locally and A_p is $f = -y^2$ locally, see Figure 36.

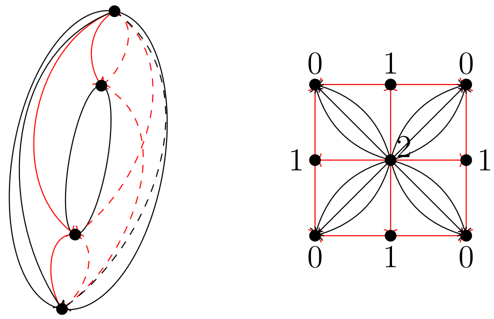


FIGURE 35. Generic geodesic flow on tilted torus and its planar view denoting the index of the critical points •

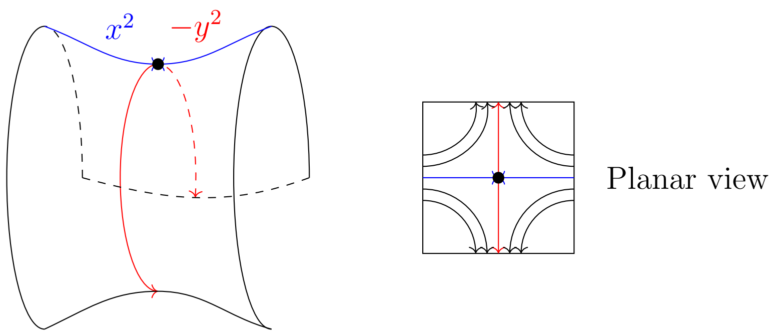


FIGURE 36. Flow lines on saddle defined by $f = x^2 - y^2$

Remark 6.13. In general, we have homeomorphism $D_p \cong (B^k)^\circ$ and $A_p \cong (B^{n-k})^\circ$ where p is the index of the critical point p.

6.2. **Palais–Smale condition and CW complex structure.** We impose that a Morse function satisfies the following genericity assumption.

Assumption 6.14 (Palais–Smale)**.** The ascending and descending manifolds all intersect transversely (in the expected dimension).

Remark 6.15. The ascending and descending manifolds for symmetric torus as in Figure 34 don't satisfy the Palais–Smale condition because the descending manifold for the top ind 1 critical point and the bottom ind 1 critical point co-incide. However, two 1-manifolds co-inciding inside 2 manifold is not a transverse intersection.

However, the tilted torus in Figure 35 satisfies the Palais–Smale assumption.

Proposition 6.16. *If a Morse function $f : M \to \mathbb{R}$ satisfies the Palais–Smale condition then the open cells $\{D_p\}$ gives rise to a CW complex structure on M.*

Example 6.17. The planar view in Figure 35 shows that D_p gives rise to a CW complex structure of a torus. Namely, D_p where p is the ind 0 critical point is p itself which corresponds to a 0-cell. When p is ind 1 critical point then the descending manifold consists of two flow lines that go to the ind 0 critical point. This gives us the 1-cells. When p is ind 2 then the descending manifold is everything else on the torus. This corresponds to a 2-cell.

Remark 6.18. If we are in non-generic situation then we don't get a CW complex structure. In fact, on a CW complex structure, the limit points of an n-cell must be in $n-1$ cell.

In the case of Figure 34, note that the closure of D_p for top ind 1 critical point contains the bottom ind 1 critical point, which lies in a 1-cell not 0-cell. Therefore, $\{D_p\}$ fails to provide a CW complex structure for a torus.

Proposition 6.19. *In the generic case, the pairs of points of adjacent index have a finite set of flow lines between them.*

Proof. In fact, suppose p be an $\operatorname{ind} k$ critical point and q be an $\operatorname{ind} k-1$ critical point, see Figure 37. Since D_p is a k-dimensional manifold, $\operatorname{codim} D_p = n-k$ and $\operatorname{codim} A_q = k-1$.

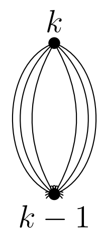


FIGURE 37. Flow lines between critical points of adjacent index

Therefore,

$$\operatorname{codim} D_p \cap A_q = (n-k) + (k-1) = n-1.$$

In particular, $\dim D_p \cap A_q = 1$. Since $D_p \cap A_q$ consists of flow lines connecting p and q, the set of flow lines, $D_p \cap A_q/(\text{translation of } \mathbb{R} \text{ on flow lines})$, is finite. □

Example 6.20. In the planar view of Figure 35, there are infinitely many flow lines connecting ind 2 and ind 0 critical points. However, there are only finitely many flow lines connecting ind 1 and ind 0 critical points and ind 2 and ind 1 critical points.

The upshot of the finiteness of flow lines between critical point of adjacent index is that we can get a homological information of the boundary map ∂ of the chain complex $C_*^f(M)$ (also known as Morse–Witten complex) of M associated to f. More precisely,

$$C_*^f(M) := \mathbb{Z}[\text{oriented critical points}]/(\overline{p} = -p)$$
$$\partial(p) := \sum_{\operatorname{ind} q = \operatorname{ind} p - 1} (\#\text{flow lines from } p \to q \text{ counted with orientation sign})q.$$

Here, orientation means choosing an orientation of the negative eigenspace of df_p (or equivalently descending manifold).

People did not appreciate this about Morse theory until Witten wrote a paper [Wit82] on Morse theory. People before Witten knew that f would give a CW decomposition. But they did not explicitly compute the degree of the attaching maps (to define the matrix element of ∂). The degree of the attaching map given by the preimages do count how many flow lines go from one critical point to the other. However no one explicitly wrote it and exploited it. After people went out and did finite dimensional Morse theory again.

This allows us to compute the homology and will allow us to study the homotopical information of a manifold.

6.3. **Handle body decomposition.** The CW complex structure via n-cells arising from a generic Morse function is good for homotopy theory but to carry out the proof of Theorem 5.7, we have to build a smooth manifold M like differential topologist via its *handle decomposition*, which is a thickened cell decomposition.

Definition 6.21. Suppose M^n is a manifold with boundary ∂M. Then *attaching a k-handle to M* means gluing on an n-ball of the form $B^k \times B^{n-k}$ along attaching map which is an embedding of half the boundary $S^{k-1} \times B^{n-k}$ into ∂M.

Remark 6.22. Recall that attaching a k-cell means attaching B^k along its boundary. Likewise, attaching a k handle amounts to attaching a thickened version $B^k \times B^{n-k}$ of B^k.

Note that the boundary satisfies the Leibniz rule:

$$\partial(B^k \times B^{n-k}) = (S^{k-1} \times B^{n-k}) \cup_{S^{k-1} \times S^{n-k-1}} (B^k \times S^{n-k-1}).$$

Example 6.23. Set $n = 2$, $k = 1$. Note that $\partial(B^1 \times B^1) = (S^0 \times B^1) \cup_{S^0 \times S^0} B^1 \times S^0$. Then attaching a 1-handle is like attaching a thickened line, see Figure 38.

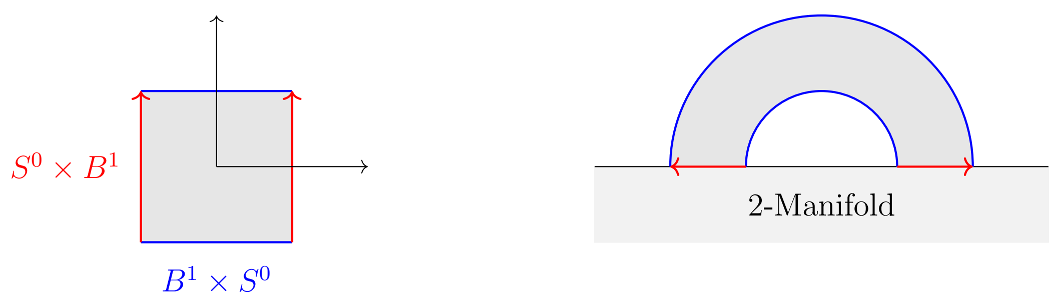


FIGURE 38. Attaching $n = 2$, $k = 1$ boundary on a 2 manifold

Example 6.24. Set $n = 2$, $k = 0$. Note that $\partial B^0 = \emptyset$. Therefore, the attaching map of 0-handle does not glue anything. In particular, attaching 0 handle amounts to taking disjoint union with disc B^2, see Figure 39.

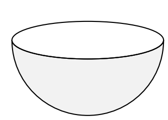


FIGURE 39. $n = 2, k = 0$ handle is B^2

Example 6.25. Set $n = 2$, $k = 2$. Then attaching 2-handle means attaching $B^2 \times B^0$ via embedding of its boundary $\partial B^2 = S^1$.

Remark 6.26. Attaching 2-handle is like attaching a 2-cell in the CW complex. The difference is that the attaching map is an embedding as opposed to any continuous map in the setting of CW complex. In particular, for n-cell attaching map, we could send $\partial B^2 = S^1$ to a point. However, 2-handle attaching map has to sent $\partial B^2 = S^1$ to an S^1 inside M. This effectively is filling in a hole (or capping off a boundary component) of a 2-manifold with boundary.

6.3.1. *Examples of handle body decomposition.*

Example 6.27 (Torus)**.** Start with empty set. First attach a 0-handle. Then attach two 1-handles so that their "feet" are inter-weaved. Then attach a 2-handle along its (red) boundary as in Figure 40

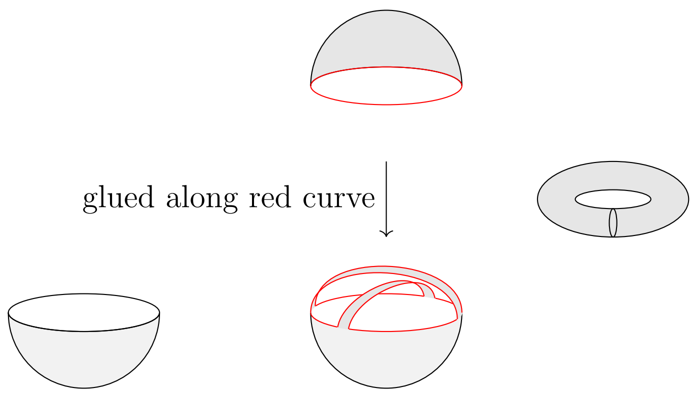


Figure 40. Handle decomposition of a torus

Example 6.28 (Sphere)**.** Start with empty set. First attach a 0-handle. Then attach two 1-handles so that their "feet" are not inter-weaved. Then attach three 2-handles to cap off the holes to get a sphere. See Figure 41.

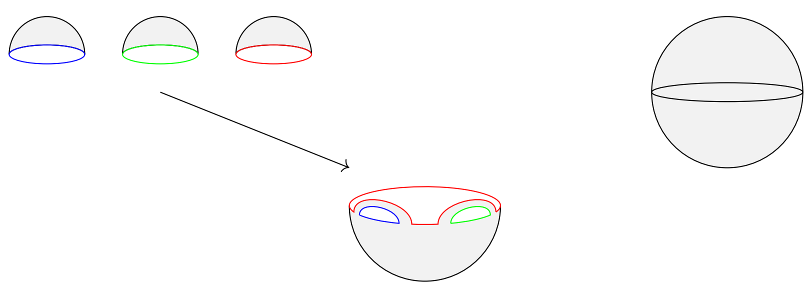


Figure 41. Attaching 2-handles on three-holed sphere

Remark 6.29. The diffeomorphism type of the result of handle decomposition depends only on the isotopy class of the embedding (attaching map).

6.4. **Morse function and handle body decomposition.** A Morse function gives a handle body decomposition of a manifold not just a CW cell complex structure.

Lemma 6.30. *If there are no critical points with critical values between a and b then $f^{-1}[a, b]$ is a product and in particular $f^{-1}\{a\} \cong f^{-1}\{b\}$.*

Proof. Use the flow lines of the Morse function and its gradient flow to get a one-to-one correspondence between two level sets $f^{-1}(a)$ and $f^{-1}(b)$. Note that these level sets are smooth $n-1$ manifolds since a and b are not critical values. See Figure 42. □

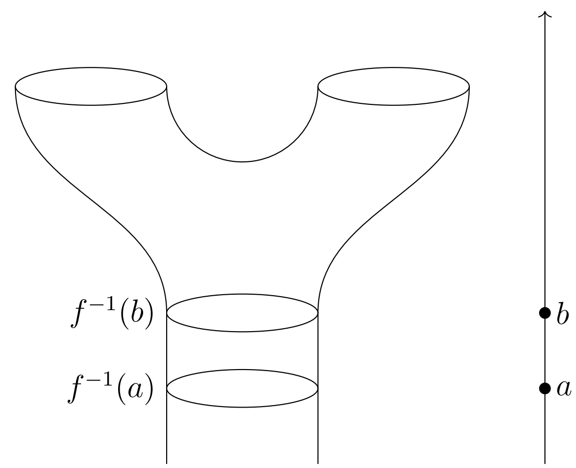


FIGURE 42. Level sets between values that are not critical are diffeomorphic.

Lemma 6.31. *If there exists a single critical point in* $[b, c]$ *then gluing* $f^{-1}[b, c]$ *is equivalent to attaching a* k*-handle to the manifold* $f^{-1}(\leq b)$*.*

Proof Sketch. Consider the schematic Figure 43 for the torus. Suppose • is an ind 1 critical

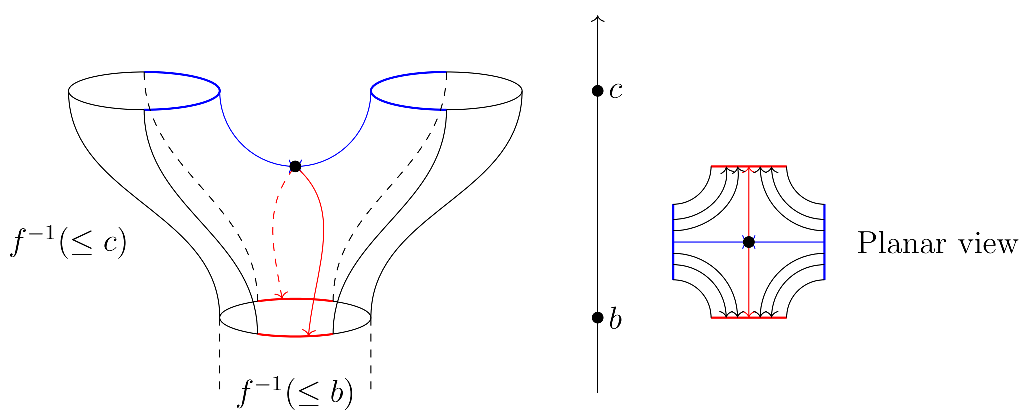


FIGURE 43. Attaching handle via Morse function and its planar view

point. Consider a central slab of $f^{-1}[b, c]$ that contains ascending and descending manifolds of p. Outside of the slab, the flow lines provide a product structure. So we can think of it as a collar neighborhood which we can ignore. On the other hand, attaching central slab along its red boundary is same as attaching a 1-handle. See the planar view of the centra slab in Figure 43. □

We summarize what we have done so far.

(1) A morse function $f : M \to \mathbb{R}$ is a smooth function with isolated non-degenerate critical points.
(2) We require the critical points to be at different heights so that one handle or cell is attached at a time, see Figure 43. More precisely if $[b, c]$ has one critical value then the topological change from $f^{-1}(\le b)$ and $f^{-1}(\le c)$ is

$$f^{-1}(\le c) \cong f^{-1}(\le b) \cup k \text{ -cell},$$

where k is the index of the critical point.
(3) We impose Palais–Smale condition on the Morse function: ascending manifolds and descending manifolds are transversal, i.e., $A_p \pitchfork D_q$. It is useful when we want to attach many handles "at the same time." Here we mean that all of the "feet" of handle getting attached will be disjoint in the attaching manifold.

Consider Figure 44. If the feet are not disjoint then one feet of the second handle we attach might land on the first handle as in Figure 44 so that the descending manifold (red) of the second overlaps with ascending manifold (blue) of the first. In such case, we don't have canonical way to isotope it down as opposed to the case in right part of Figure 44. In fact, if a foot lands on the ascending manifold of of the

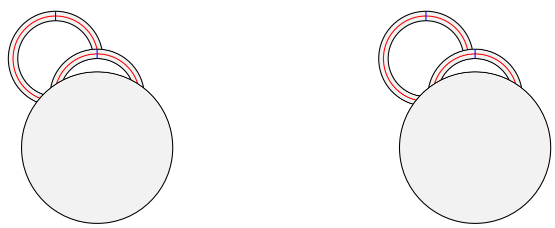


FIGURE 44. Attaching handles at different times vs at "same time"

critical point of the same index then we get non-isotopic embedding. However, we can make a hypothesis that we consider isotopy up to handle slide, defined below.

Definition 6.32. *Handle slide* is the motion of "foot" of a k-handle over another (Figure 45).

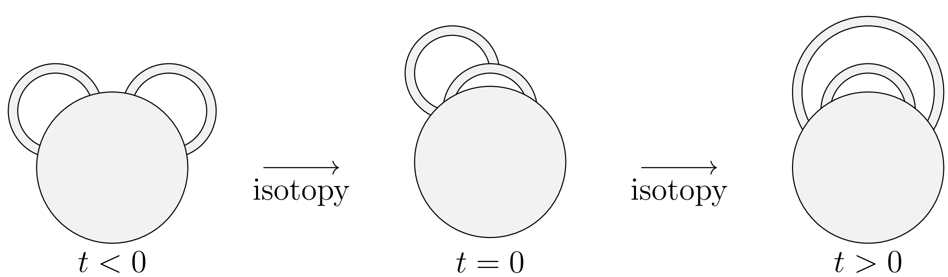


FIGURE 45. Handle-slide

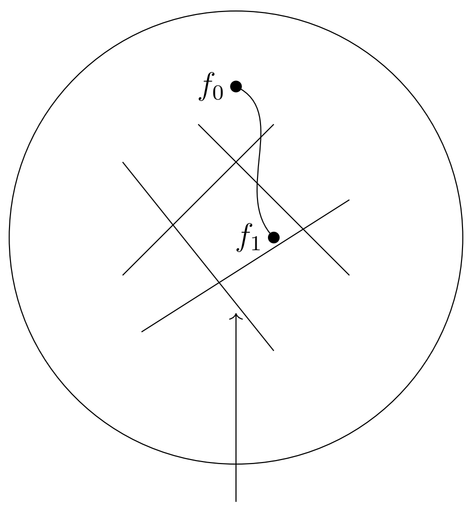
Remark 6.33. We consider handle body attachment up to isotopy and handle slides. Nevertheless, we can think of handle slide and isotopy after attaching one handle.

6.5. **Families of Morse function.** Now that we have a handle body decomposition corresponding to a generic Morse function, we need to make sure that assigning Frobenius algebra structure to two different handle body decomposition corresponding to two generic Morse functions result in the same algebra. In particular, the two handle body decomposition should be related via Frobenius relations only. To do so, we have to study the space of smooth functions and generic path between Morse functions and the change in handle body decomposition along the path.

Before we state our theorem let's see what can happen

One parameter family of morse function.

Space of smooth functions



Driscriminant={non-Morse-functions}

Figure 46. Schematic picture of the space of smooth functions where generic points f_i are Morse functions while the discriminant consists of non-Morse functions

Singularity theory analyzes what happens in the generic path of Morse functions. The main complication is that the space of Morse function is probably disconnected because of the presence of non-morse functions. However, the space of smooth function is path connected, namely, linear interpolation takes one Morse function f_1 to another Morse function f_2 but not in a generic way, i.e., a path that is transverse to the discriminant. To understand generic paths, we have to understand the discriminant. Note that a function in the disrciminant can be very bad for instant the constant function 0 is so far from its codimension infinity. It has infinitely many conditions on the functions to make it identically zero.

The following theorem says that the intersection points of generic path between two Morse function and the discriminant are at the simplest kind of non-morse function that can happen.

Theorem 6.34. *A generic* 1*-parameter family of smooth functions* $\{f_t\} : I \times M \to R$ *which are Morse function (with all three properties) except at isolated times, where one of the three things happen:*

(1) A function with critical points at equal height.
(2) A function with ascending manifold not transverse to a descending manifold, resulting in a handle slide.
(3) A birth or death point, i.e., locally the one parameter family of smooths functions have a cubic term in the Taylor expansion:

$$f_t(x) = (x_1^3 - tx_1) + [-\sum_{i=2}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2].$$

Example 6.35 (Cross-over)**.** Fix $n = 2$. Consider a family of Morse on three-legged pants and two Morse functions as shown in $t < 0$ and $t > 0$ so that critical points are at different heights. Then a smooth interpolation between $f_{<0}$ and $f_(> 0)$ can pass through a function so that the critical points of ind 1 have same height, see Figure 47. This is called *cross-over*. It corresponds to the associativity relation in the Frobenius algebra.

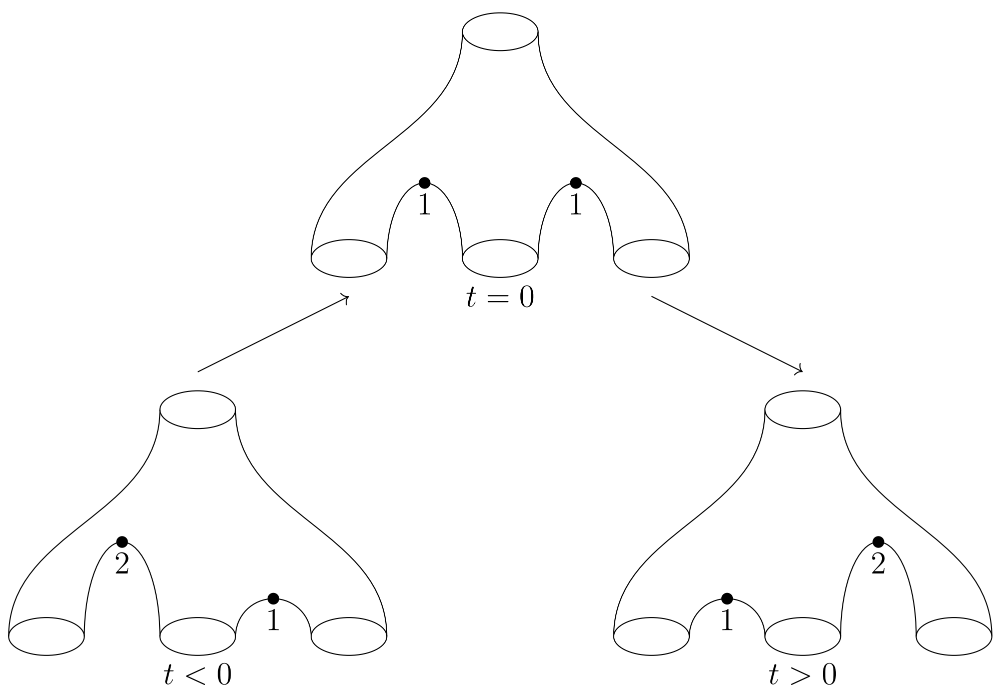


FIGURE 47. Cross-over of index of critical points

Example 6.36 (Birth-death point)**.** Set $n = 2$. In the birth process, when $t = 0$, there is an inflection point of f_0. When t is positive the first term is cubic with maximum and a minimum but when it is negative there is no minimum or maximum or inflection point, see Figure 48.

In general birth process, going from $t < 0$ to $t > 0$ results in creation of two critical points of adjacent index, i.e, a maximum of ind k and a minimum of ind $k-1$. There is a unique flow between them. There is also an upside down version of Figure 48 corresponding to death process.

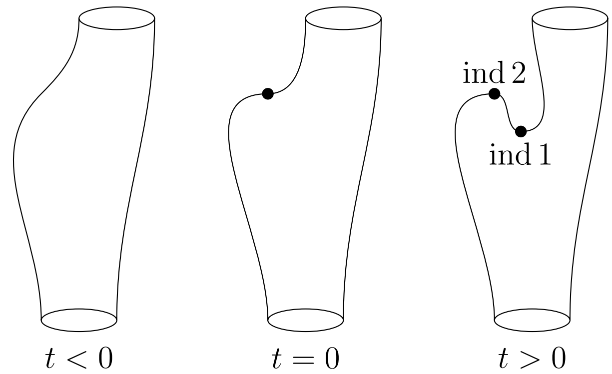


FIGURE 48. Birth-death phenomena of crictical points of Morse functions in $n = 2$. When $t < 0$, there is no critical point. When $t = 0$, there is one degenerate critical point. When $t > 0$ there are two non-degenerate critical points of adjacent indices 2 and 1.

This process in Frobenius algebra structure corresponds to the co-unital structure, Figure 22.

A generic one parameter family connecting two Morse functions f_1 and f_2 can be described by a "graphic:=" graph of heights of critical point versus time t as in Figure 49. In the graph,

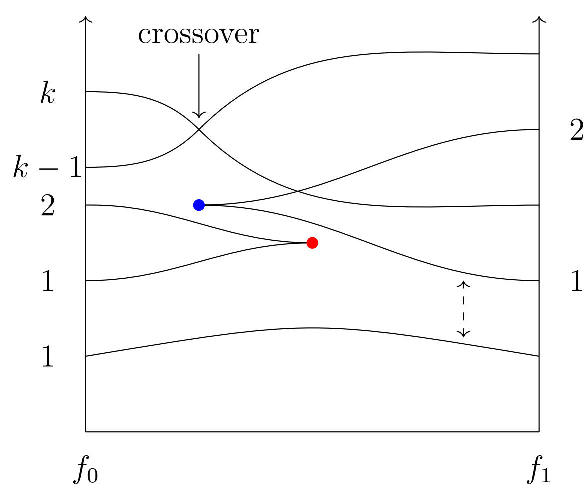


FIGURE 49. Graph of heights of critical points of a family of functions $\{f_t\}_{t\in[0,1]}$. $\bullet$ is a death process $\bullet$ is a birth process. $\leftarrow\text{-----}\rightarrow$ is a handle slide.

the numbers represent the indices of critical point at the corresponding height. Cusp at the right side represents death process. Cusp at the left side represents birth process. The dotted double sided arrow represents handle slide.

Hint to proof of Theorem 5.7. In [HT80], Hatcher and Thurston give a presentation of a mapping class group of a surface. Along the way, they analyzed the theorem in detail for

$n = 2$. Namely, they prove that the "moves" that result are the Frobenius relations for Frobenius algebra. For a complete proof see [Koc04]. It is a 250 pages long book. □

The upshot is that we have a surface as a cobordism between some number of circles to other number of circles in Figure 50. We put a generic Morse function on the surface to get a level set decomposition giving rise to a handle-body decomposition. A Morse function gives a factorization of a cobordism surface into elementary pieces. And for different factorization, we can form a generic one-parameter family of factorization the equivalent of which correspond to structural relations in the Frobenius algebra.

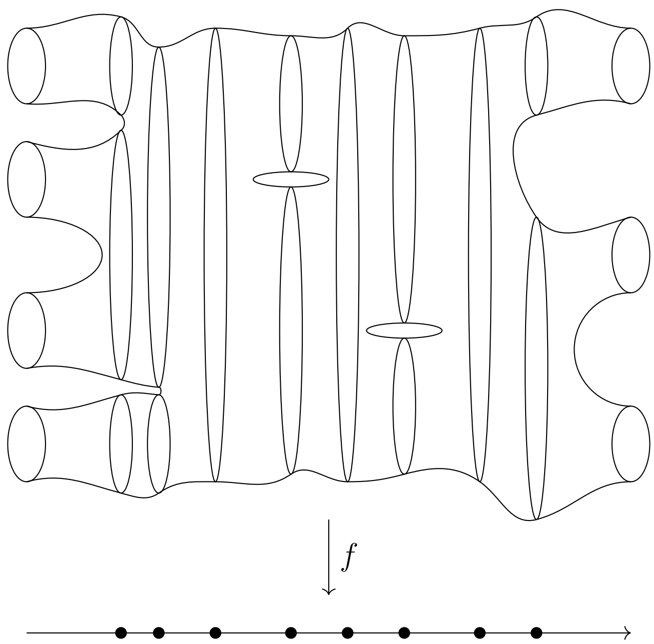


FIGURE 50. Factorization of surface via a Morse function f. The level set of interval between two adjacent • contains a critical point of f.

Remark 6.37. We have swept under the rug the relation corresponding to symmetry (or permutation of boundary circles) (Figure 18). It does not arise from Morse theory. However, with a lot of categorical theoretic philosophical fiddling around we can relate symmetry at a topological level and Frobenius level. See [Koc04] for more details.

7. Knot polynomials before TQFT

Knot polynomials emerged before TQFT was defined but they illustrate the TQFT structure very well. In this §we will talk about knot polynomials and the work before Witten. This section will motivates us to see what Witten actually did.

7.1. **Knot group.** The first knot polynomial was defiend by Alexander in 1928 [Ale28]. The Alexander polynomial is classical topology measuring something quite natural.

Definition 7.1. A *knot* $K \subset S^3$ is an embedded S^1 up to ambient isotopy.

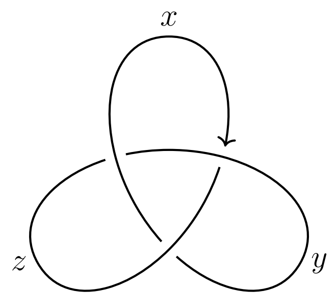


FIGURE 51. Trefoil

The knot exterior $X_k \coloneqq S^3 - N^\circ(K)$, where $N^\circ(K)$ is an open neighborhood of K in S^3. Note that $X_k \simeq C_k = S^3 - K$, which is a 3-manifold with a hole in it in some sense. Further, $\partial X_k = S^1 \times S^1$.

The "simplest" thing would be the knot group $\pi_1(X_k)$. Note that $H_*(S^3 - K) \cong H_*(S^1)$ follows from Alexander duality. In particular, $H_1(X_k) = \mathbb{Z}$, which is the abelianization of $\pi_1(X_k)$. Therefore, first, the homology group of X_k are boring. On the other hand, $\pi_1(X_k)$ has to be infinite making it very complicated to study.

A diagram D of a knot gives a finite presentation of $\pi_1(X_k)$ via Weirtinger presentation. For instance, from the diagram in Figure 51 of the trefoil, we can infer that

$$\pi_1(X_k) \coloneqq \langle x, y, z \mid z^x = y, x^y = z, y^z = x \rangle.$$

Here $z^x \coloneqq x^{-1}zx$ is the conjugation.

On the other hand, there's no algorithm for deciding whether two finite presentations of infinite discrete groups are isomorphic, making it harder to study knots via $\pi_1(X_k)$.

So we will study $\pi_1(X_K)$ by simplifying it via the covering spaces.

7.2. **Seifert Surfaces and covering of knot complement.** Alexander realized that the homomorphism (via abelianization)

$$\phi : \pi_1(S^3 - K) \twoheadrightarrow \mathbb{Z} = \langle t \rangle$$

defines an infinite cyclic cover $\widetilde{X_k}$ with the deck transformation group $\mathbb{Z} = \langle t \rangle$.

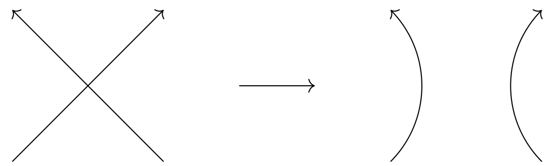
Remark 7.2. In general, there is a covering space corresponding to a subgroup of $\pi_1(X_k)$ or equivalently corresponding to the $\ker \phi$.

To picture $\widetilde{X_k}$, we first choose a *Seifert surface* of K.

Definition 7.3. A *Seifert surface* is a connected oriented surface Σ with $\partial\Sigma = K$ after orienting K.

The existence of a Seifert surface can be guaranteed via Seifert's algorithm:

(1) Smooth out any crossing in a knot diagram via the map:



The result will be a collection of unknots (the standard S^1).

(2) To each unknot, bound a disc.
(3) Reconnect the discs via half-twisted bands connecting the regions where there was a crossing in the first step.

The result is a Seifert surface.

Example 7.4. For the trefoil as in Figure 51, Step 1 gives two unknots with the same orientation. Then filling in the unknots and attaching three twisted bands we get the Seifert surface in Figure 52. Notice that the surface is oriented.

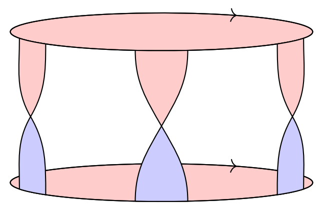


FIGURE 52. Seifert's algorithm to create Seifert surface of Trefoil in Figure 51

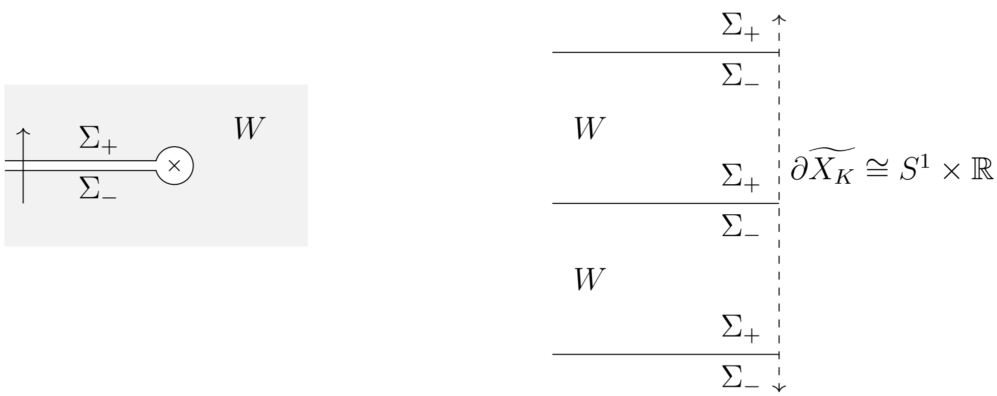


FIGURE 53. Left: End on view of the knot complement X_K cut along the Seifert surface Σ (with orientation). K is going into the page at the cross mark. Right: Schematic view of $\widetilde{X_K}$.

Once we have a Seifert surface, we can schematically think of the covering map $\widetilde{X_K} \to X_K$ like the infinite cyclic covering of $\mathbb{C} \setminus 0$ at least in the end-on-view of K. In that case, ϕ is like the winding number function. Note that X_K from end-on view of K looks like $\mathbb{C}$

minus a neighborhood of the origin. If we cut X_K along Σ then we get a 3-manifold W with boundary

$$\partial W = \Sigma_- \cup S^1 \times I \cup \Sigma_+.$$

Schematically, W looks like $\mathbb{C}$ minus the neighborhood of non-positive half real line, see Figure 53. Then we can stack countably many copies of W and glue along $\Sigma_\pm$ with appropriate gluing as in Figure Figure 53 to get $\widetilde{X_K}$. The deck transformation is given by shift say t.

7.3. **Alexander module and polynomials.**

Definition 7.5. The *Alexander module* is $H_1(\widetilde{X_k};\mathbb{Z})$ viewed as a module over group ring $\mathbb{Z}[t^\pm]$ where the action of $\mathbb{Z}[t^\pm]$ is an extension of the action of deck transformations $\mathbb{Z} = \langle t\rangle$ of the covering $\widetilde{X_k} \to X_k$.

Using Mayer–Vietoris on the description of $\widetilde{X_K}$ as covering space in Figure 53, we can compute its homology. Moreover, tracking the boundary maps in the MV, we can prove that a presentation matrix for $H_1(\tilde{X}_k;\mathbb{Z})$ as a $\mathbb{Z}[t^{\pm 1}]$ module is $tA - A^T$, where A is the *Seifert matrix.* More precisely, if g is genus of the Seifert surface, then A is $2g \times 2g$. Further, we have a presentation of the Alexander module (as a part of the Mayer–Vietoris sequence):

$$\mathbb{Z}[t^\pm]^{2g} \overset{tA-A^T}{\to} \mathbb{Z}[t^\pm]^{2g} \to H_1(\tilde{X}_k) \to 0. \tag{7.1}$$

See [Lic97, Theorem 6.5] for more details.

We can compute the Seifert matrix A from the Seifert surface Σ in the following way. Choose a basis $\{e_i\}$ of $H_1(\Sigma)$ and compute the linking number $\mathrm{lk}(e_i, e_j^+)$ where e_j^+ is e_j pushed into X_k in the positive direction defined by the right hand rule and orientation of the surface.

Definition 7.6. Let D be a diagram for two disjoint oriented knots $K_1, K_2 \subset \mathbb{S}^3$. The *linking number* of K_1 and K_2 is

$$\mathrm{lk}(K_1, K_2) \coloneqq \frac{1}{2}\sum_c \mathrm{sgn}(c)$$

where c runs over crossings in D between K_1 and K_2 with the convention that $\mathrm{sgn}(c_+) = 1$ and $\mathrm{sgn}(c_-) = -1$ where c_+, c_- are crossing shown in the Figure 54.

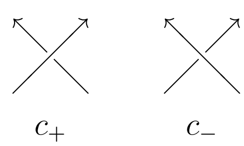


FIGURE 54. Crossings

Example 7.7. We compute the Seifert matrix for the trefoil, see Figure 55.

Then using the definition of linking number and Figure 55 we can see that

$$\mathrm{lk}(e_1, e_1^+) = 1, \quad \mathrm{lk}(e_2, e_2^+) = 1, \quad \mathrm{lk}(e_1, e_2^+) = 0, \quad \mathrm{lk}(e_2, e_1^+) = 1.$$

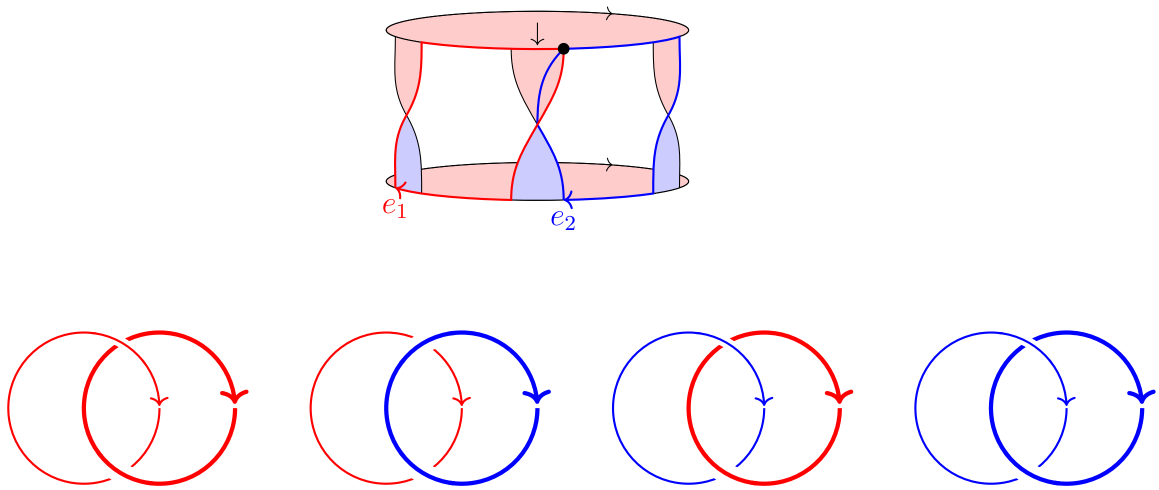


FIGURE 55. Top: Seifert's Surface of the Trefoilin Figure 51 with generators. The positive direction is defined by the right hand rule and the orientation of the boundary. Bottom: Thickened red circle is e_1^+ and thickened blue circle is e_2^+.

Therefore,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7.2}$$

Remark 7.8. Since $\mathbb{Z}[t^{\pm}]$ is not a PID, modules over $\mathbb{Z}[t^{\pm}]$ are complicated to study. In fact, we don't have a classification theorem for them. Therefore, studying Alexander module is quite complicated.

Fact 7.9. Nevertheless, $\det(tA - A^T)$ is a well-defined invariant up to multiplication by units in $\mathbb{Z}[t^{\pm 1}]$, i.e., $\pm t^n$.

Remark 7.10. The reason for indeterminacy of the determinant up to multiplication is that the presentation matrix is not an invariant of the module it represents. However, it is unique up to row and column operation and stabilization operation. For instance, we could change the basis.

Definition 7.11. The Alexander polynomial $\Delta_K(t)$ of a knot K is $\det(tA - A^T) \in \mathbb{Z}[t^{\pm 1}]$.

Example 7.12. For the trefoil, the Alexander polynomial can be computed using (7.2):

$$\det(tA - A^T) = t^2 - t + 1.$$

Remark 7.13. There are other invariants we can extract from the Alexander module.

Fact 7.14. (1) For a knot, Δ satisfies

$$\Delta_K(t = 1) = \pm 1.$$

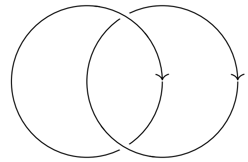


FIGURE 56. Hopf link

(2) For a link (that is not a knot), there still exists a canonical (abelianization) map $\phi : \pi_1(X_L) \twoheadrightarrow \mathbb{Z}$ given by sending the meridians of the components to the generator. We can still prove the existence of a Seifert surface. However,

$$\Delta_L(t = 1) = 0.$$

An example of link is the Hopf link:

(3) $\Delta(t) = \Delta(t^{-1})$ up to $\pm t^n$.

Remark 7.15. (1) Traditionally people normalized the Alexander polynomial for knots by setting

$$\Delta(t) = \Delta(t^{-1})$$
$$\Delta_K(1) = +1.$$

This removes the ambiguity of multiplication.

(2) However, the above normalization does NOT work for links. This held up progress for long time. The general theory of Alexander polynomials was invented in 1928 involving 3-manifolds and their topological information. However, the most important formula (Skein relations, see below) for Alexander polynomial took a long time to be discovered.

7.4. **Knot polynomials via Skein relations.** Conway in [Con70] discovered that

$$\Delta_L(t) := \det(t^{1/2}A - t^{-1/2}A^T)$$

is well-defined on the nose even for links and it satisfies the *Skein relations*:

$$\Delta_{L_+} - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}, \tag{7.3}$$

where L_+, L_- and L_0 are links that differ locally by change of crossings as shown in Figure 57.

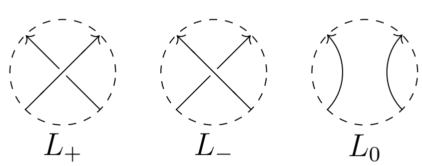


FIGURE 57. Local pictures of links differing by change of crossings

Remark 7.16. Since there is ambiguity in $\pm$ sign for Alexander polynomials, the Skein relation only makes sense once the polynomials have been correctly normalized because this is an "additive relation" for links. In contrast, there are multiplicative relations that make sense up to multiplication: if we take connect sum of links then the Alexander polynomials multiply.

The Skein relations were discovered after realizing the correct normalization. There were more skein relations discovered later:relation among invariants of links that differ locally.

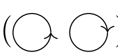
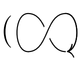
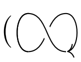
Theorem 7.17. *There exists a unique invariant called the Conway potential function $\nabla \in \mathbb{Z}[z]$ of oriented links satisfying*

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0),$$
$$\nabla(\bigcirc) = 1.$$

We should think $z = t^{-1/2} - t^{1/2}$.

Proof. Existence comes from Seifert matrix method. Uniqueness is via "complexity" (of knots and links) argument and induction. Namely, we find places where we can change crossing to later decrease the number of crossings until we get decrease complexity (have fewer crossings) and end up with bunch of unknots for which we know ∇. $\square$

Example 7.18. Using the Skein relation (7.3), we can see that

$z\nabla($  $) = \nabla($  $) - \nabla($  $) = 0.$

Therefore, the Conway potential for two unknots is 0.

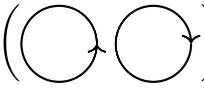
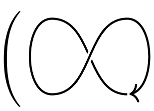
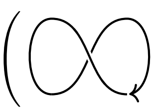
Remark 7.19. Nobody really exploited the Skein relations until 1985.

Theorem 7.20 (Jones 1984)**.** *There exists a unique invariant $V(L) \in \mathbb{Z}[t^{\pm 1/2}]$ of links such that*

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$$
$$V(\bigcirc) = 1.$$

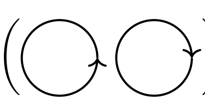
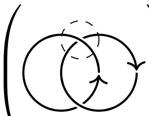
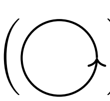
Remark 7.21. Jones was actually studying operator algebra. He related some of his problems to problems about braid groups which he later turned to problems of knots.

Example 7.22. (1) Note that

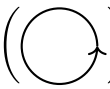
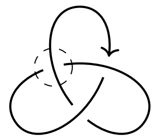
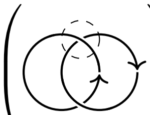
$(t^{1/2} - t^{-1/2})V\Big($  $\Big) = t^{-1}V\Big($  $\Big) - tV\Big($  $\Big)$

$= t^{-1} - t.$

(2) To compute the Jones polynomial of negative Hopf link note that

$t^{-1}V\Big($  $\Big) - tV\Big($  $\Big) = (t^{1/2} - t^{-1/2})V\Big($  $\Big).$

(3) To compute the Jones polynomial of trefoil knot note that

$t^{-1}V\Big($  $\Big) - tV\Big($  $\Big) = (t^{1/2} - t^{-1/2})V\Big($  $\Big).$

Theorem 7.23 (HOMFLY–PT)**.** *There exists a unique polynomial invariant $P \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ such that*

$$a^{-1}P(L_+) - aP(L_-) = zP(L_0)$$
$$P(\bigcirc) = 1$$

simultaneously generalizing Conway/Alexander and Jones.

Remark 7.24. In a sense, HOMFLY-PT polynomial is a completely general polynomial invariant. Namely, there is a three variable (but homogeneous) polrnomial

$$xP(L_+) + yP(L_-) + zP(L_0) = 0.$$

When we remove homogeneity, we get a two variable polynomials.

In the rest of this §we will outline the proof of Theorem 7.20.

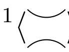
The uniqueness via a complexity argument. Complexity can be captured by the number of crossing in the diagrams and the number of crossings it takes to get to the unknots.

There are many proofs in the literature on the existence of the Jones polynomials. We follow earliest methods via the Kauffman's bracket.

Theorem 7.25 (Existence of the Kauffman bracket)**.** *There exists a unique function called the Kauffman's bracket*

$$\langle \cdot \rangle : \{\textit{unoriented links}\} \to \mathbb{Z}[A^{\pm}]$$

such that

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$$\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$$
$$\langle \bigcirc \rangle = 1.$$

Proof. The proof of the theorem is immediate. □

Exercise 3. Compute the Kaufmann bracket of the trefoil.

Theorem 7.26 (Reidemeister's theorem)**.** *Two oriented links L and L' are equivalent if and only if any two of their diagrams are related by a sequence of oriented planar isotopies and the Reidemeister moves shown in Figure 58.*

Proof. The proof of this theorem amounts to analyzing the singularities of projections as we isotope a link in 3-space. We could do so by assuming that every link is piecewise linear. Then it boils down to understanding how triangles move (?) □

Corollary 7.27. *If a function is invariant under the Reidemeister moves then it is a link invariant.*

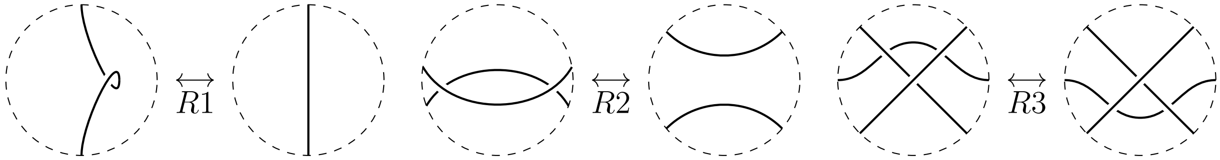
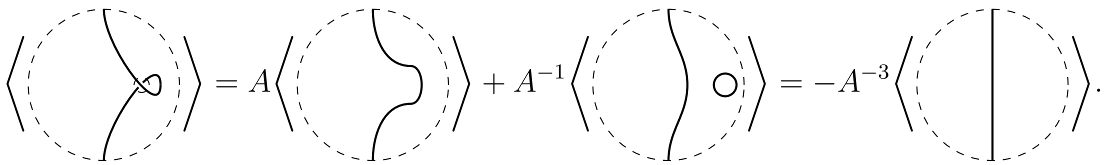


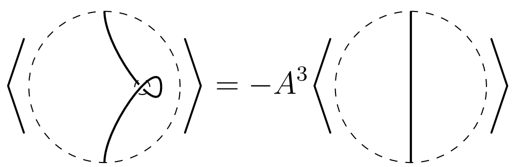
FIGURE 58. Reidemister Moves

Lemma 7.28. *Kaufmann bracket is invariant under the Reidemeister moves $R2$ and $R3$ and is multiplication by $-A^{\pm 3}$ under $R1$.*

Proof. Note that

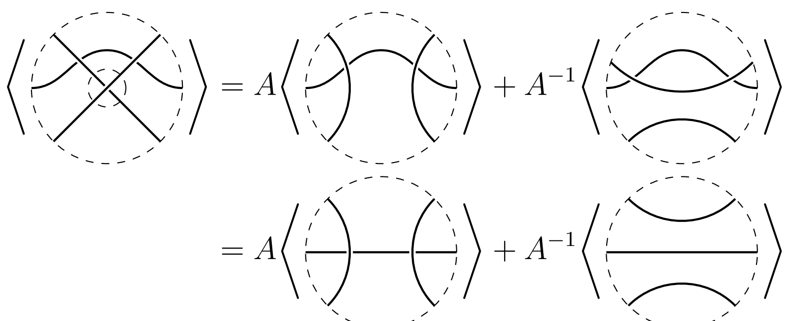


Similarly, we have

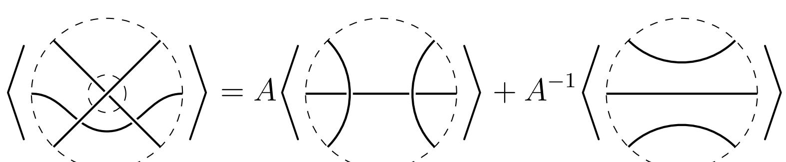


This proves the last statement.

Since the computation is fairly similar, we will only prove the invariance of the Kauffman bracket under $R3$.



By rotating the picture by 180 degree, we see that



Therefore, $\langle \cdot \rangle$ is preserved under $R3$. $\square$

One way to fix that $\langle , \rangle$ is not invariant under R_1 is to use *writhe* of oriented diagonal.

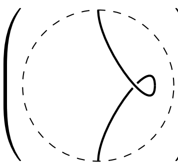
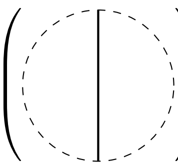
Definition 7.29. A *writhe* of a link with oriented diagram D is

$$w(D) \coloneqq \sum_c \operatorname{sgn} c,$$

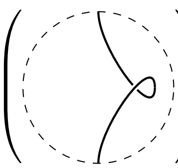
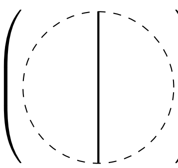
where $\operatorname{sgn}(c_+) = 1$ and $\operatorname{sgn}(c_-) = -1$ where $c_\pm$ are defined in Figure 54.

Fact 7.30. Check that writhe is invariant under $R2$ and $R3$. However, it is neither a knot nor a link invariant.

Observe that

w () $= w$ () $- 1$

regardless of of the orientation. Further,

w () $= w$ () $+ 1.$

Writhe and Kauffmann brackets are two non-invariants of the same fashion. We could use both of them to define a link invariant such that for an oriented link with diagram D

$$f(D) \coloneqq (-A^3)^{-w(D)} \langle D \rangle.$$

Theorem 7.31. *Setting $A = t^{-1/4}$ turns $f(D)(A)$ into the Jones polynomial.*

Proof. Simply compute the Skein relations. Jones polynomial is characterized by the Skein relations. All we have to do is compute f for $L_0, L_\pm$. ☐

Question 7.32. Is there a bound on the number of Reidemeister moves from one knot to another?

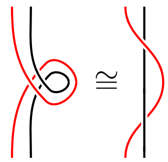
Is there a function of N such that for diagrams with at most N crossings, the maximal length of a Reidemeister moves needed to related them is the function? This might give an algorithm of classifying knots.

7.4.1. *Kauffmann Brackets and framed knots.* An alternative way of fixing the bracket without using writhe is to consider *framed links.*

Definition 7.33. A *framed link* is a link with a choice of non-vanishing normal vector fields, considered up to homotopy.

Remark 7.34. There is a $\mathbb{Z}$-indexed (linking number) family of framings for each components. It corresponds to the times the framing loops around the diagram of the link

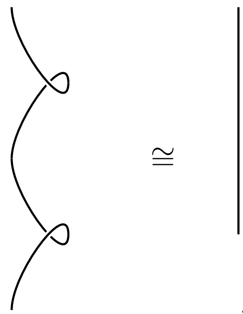
It's natural to use diagram where "blackboard framing" is assumed. For instance,



Theorem 7.35. *There is a correspondence*

$$\{\text{link diagrams modulo}/R_2, R_3\} \cong \{\text{isomorphism classes of framed links}\}.$$

Remark 7.36. (1) The above theorem says that if don't use Reidemester move 1 then we get framed links and nothing more than that. Further,



In particular, we can simply accept the Kauffmann's bracket as an invariant of framed links, not ordinary one. An advantage of doing so is that we don't have to worry about orientation.

(2) Although correction by writhe works, in practice it is convenient to do framed links. When we come to 3 manifold invariants, we need framing of links. It is actually good that need a framing to be well defined.

7.5. **Some properties of Jones polynomials.**

Theorem 7.37. *The Jones polynomials satisfies the following properties.*

(1) If $\overline{L}$ is the mirror image of L obtained by reversing all the crossings of L, then

$$V(\overline{L})(t) = V(L)(t^{-1})$$

(2) Denoting $\#$ to be the connect sum of links,

$$V(K_1 \# K_2) = V(K_1)V(K_2).$$

(3) Denoting breadth to be the spread of powers of t of the Laurent polynomials then

$$\text{Breadth} V(L) \leq \#\text{of crossing of } D.$$

The equality holds for reduced alternating diagrams.[3]

An unreduced diagram has isthmus.

Remark 7.38. Alexander polynomials does not satisfy the first property. In particular, it does not detect mirror images of a link. In fact, the universal cover of $X_{\overline{L}}$ is the same as that of X_L with orientation reversed so the homology group will be the same. It turns out

[3]This was known as Tait's conjecture from 1890. Trefoil has an alternating diagram. A trefoil with a kink does not have an alternating diagram.

that the quantum invariants conjugate under mirror images. We can detect left-handed or right handed trefoil.

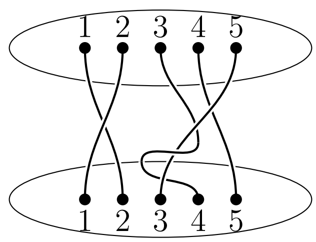
8. Knot polynomials after TQFT

We will explain where the knot polynomials "idealogically come from." Chronologically, they arose as combinatorial manipulations of invariants of knots and Seifert surfaces and their exterior. However, they fit into a general scheme of quantum invariants of knots and links which are related to quantum groups which are not really groups but Hopf algebras. We will skip explicitly mentioning quantum groups for now but demonstrate how knot invariants arise from representation of braid groups. The representation of braid groups resembles $(0+1)$ TQFT.

8.1. **Braid groups.** In the next two subsections, we will study algebraic structures related to knots and links. Traditionally, algebraic structure related to knots and links is the braid group. The second one is the Temperly–Lieb algebra. These tools are used to prove the existence of invariants of 3 manifolds.

Definition 8.1. An *n-braid* is a disjoint union of intervals on $\sqcup_{i=1}^{n} I \to \mathbb{R}^2 \times I$ (with suitable boundary) each embedded monotonically in the I-direction.

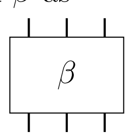
Example 8.2. In the picture below, we show a braid with $n = 5$ points.



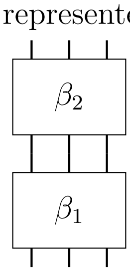
Remark 8.3. (1) We consider these braids up to isotropy preserving the I coordinate.

(2) Monotonicity rules out backtracking  and thus critical points of an embedded inteval. In fact, every slice will intersect braid with n-points.

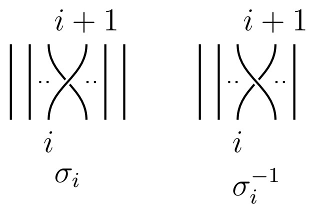
Schematically, we represent an n-braid β as



Remark 8.4. Under vertical stacking, braids form a group. The vertical stacking of two n-braids β_1 and β_2 can be schematically represented as



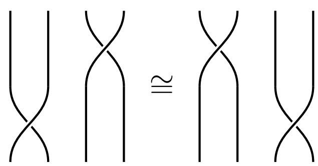
8.1.1. *Generators and relations of braid group.* Just like what we did with $\mathbf{Cob}_{1+1}$, we can use Morse theoretic arguments to factorize any braids into elementary braids which just consists of one point moving behind each other at the i-th and $i+1$-th points, i.e., a generator σ_i can be pictured as



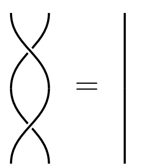
Writing $\sigma_j \leftrightarrow \sigma_j$ to mean that σ_i commutes with σ_j we have the following presentation of the braid group.

$$\langle \sigma_1 \ldots, \sigma_{n-1} \colon \sigma_i \leftrightarrow \sigma_j \text{ if } |i-j|>1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle. \tag{8.1}$$

To see the commutativity note that



The fact that $\sigma_i\sigma_i^{-1} = \mathrm{id}$ corresponds to the second Reidemeister move. In fact,



Similarly, the $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ corresponds to Reidemeister III. In fact

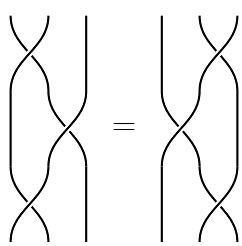


FIGURE 59. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

8.1.2. *Alternative description of Braid group.* An alternative description of the braid group is the following.

Let $\tilde{C}_n$ denote the configuration space of n distinct points in $\mathbb{C}$, i.e.,

$$\tilde{C}_n \coloneqq \{(z_1, \ldots, z_n) \colon z_i \neq z_j, \quad \forall i, j\}.$$

Equivalently, $\tilde{C}_n$ is moduli space of distinguished ordered points.

Note that the symmetric group S_n acts on $\tilde{C}_n$ by permutation of coordinates. Then we can define the quotient

$$C_n \coloneqq \tilde{C}_n / S_n$$

as the moduli space of n-tuples of indistinguishable points in $\mathbb{C}$.

Then the n-braid group B_n is defined as

$$B_n = \pi_1(C_n \text{ with respect to base points } \{1, 2, \ldots, n\} \subset \mathbb{C}).$$

An intuitive way to think about this is that a path in C_n can be viewed as a n-tuple of points times an interval moving around in space so that their end points correpsond to the same n-tuple of points at final time. Each of such loop correspond to a braid.

Remark 8.5. (1) There exists a homomorphism $\theta : B_n \to S_n$ namely, consider the induced permutation of points. In Example 8.2, the braid corresponds to (12)(354).

(2) We could consider $S_n =$ given by braided pictures with crossings considered irrelevant. Then

$$S_n = \langle \tau_1 \ldots \tau_{n-1} : \text{ same relations for braid but also } \tau_i^2 = 1 \rangle.$$

The punch line here is

Braid group is the quantization of the symmetric group.

The kernel of θ is called the *pure braid group* $P_n = \pi_1(\tilde{C}_n)$. A pure braid sends i back to i.

8.1.3. *Another alternative description.* A second alternative of thinking of B_n in terms of the mapping class group as opposed to the fundamental group of the configuration space is the following:

$$B_n = \pi_0 \operatorname{Diff}^+(B^2, \partial, \text{fixing the base set of } n \text{ points setwise}).$$

This is the set of orientation preserving diffeomorphism and is identity on the boundary and preservers the set of n points setwise. Namely, B_n is the mapping class group of disc with n-punctures relative its boundary.

Assume that all punctures are along the real axis of $D^2 \subset \mathbb{R}^2$. A generator σ_i corresponds to the diffeomorphism that rotates by 180 the inner (dashed) boundary of annulus containing ith and i-th punctures, leaves the outer dashed boundary fixed and interpolates the concentric circles in between, see Figure 60. More precisely, we do half-Dehn twist on the annulus.

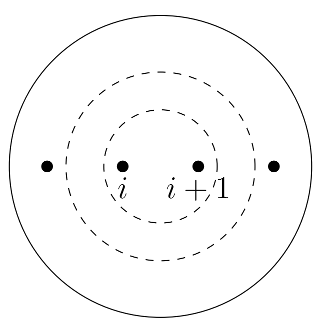


FIGURE 60. Generator σ_i of B_n

Note that there exists a fibration

$$\begin{tikzcd}
\mathrm{Diff}^+(B^2, \partial, \text{fixing the base set of } n \text{ points setwise}) \arrow[r] & \mathrm{Diff}^+(B^2, \partial) \cong * \arrow[d, "\text{look at image of base set of points}"] \\
 & C_n(B^2)
\end{tikzcd}$$

The downward map sends a diffeomorphism to its image of the n-point set.

Since $\mathrm{Diff}^+(B^2, \partial)$ is contractible, the long exact sequence of the above fibration implies that

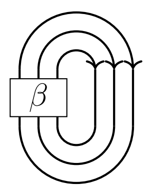
$$\pi_1(C_n(B^2)) \cong \pi_0(\mathrm{Diff}^+(B^2, \partial, \text{fixing the base set of } n \text{ points setwise})),$$

which relates the two definitions of the braid group.

8.2. **Braids and links.** There exists a closure map

$$B_n \to \{\text{oriented links}\}$$
$$\beta \mapsto \hat{\beta}.$$

Pictorially, the closure of β is



The number of component of links depends on the cycle type of the permutation that the braid defines.

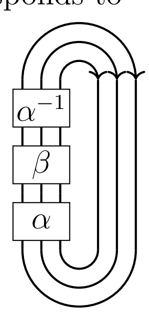
Theorem 8.6 (Alexander). *Every link is a closure of some braid in B_n (for some n).*

Remark 8.7. The minimal such n is called "braid index."

Theorem 8.8 (Markov: Reidemeister theorem for braids). *The relation on braids corresponding to isotopy equivalence of their closure is generated by two things:*

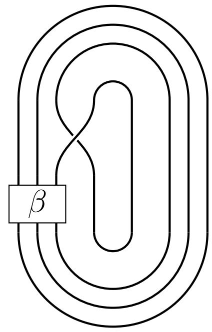
(1) Conjugation in B_n: $\widehat{\alpha^{-1}\beta\alpha} = \hat{\beta}$ for all α.
(2) Stabilization: If $\beta \in B_n$ then $\hat{\beta} = \widehat{(\beta\sigma_n)} \in B_{n+1}$. Namely, $B_n \to B_{n+1}$ sends $\beta \mapsto \beta\sigma_n$ (this map is not a homomorphism).

Pictorially, conjugations $\widehat{\alpha^{-1}\beta\alpha}$ corresponds to



Note that we can slide around α^{-1} in the clock-wise direction and compose with α to get id and thus $\widehat{\alpha^{-1}\beta\alpha} = \widehat{\beta}$.

Similarly, $\widehat{(\beta\sigma_n)}$ is pictorially represented as



Then using Reidemeister 1 move we an see that $\widehat{(\beta\sigma_n)} = \widehat{\beta}$.

Remark 8.9. Viewing links as braids allows us to study oriented links via braid groups, which is more algebraic. This allows us to find invariants of links using algebraic structures of the braid group. To find such invariants, we look for functions on braid groups that are invariant under conjugation and Markov stabilization.

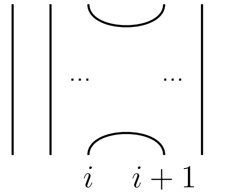
Functions that are invariant under conjugations are easy to look for. We might want to look in representations of the braid group and taking their trace (to get character) of the representation because trace of conjugate is same as the trace of the unconjugate. Namely, $\operatorname{tr}(aba^{-1}) = \operatorname{tr} b$. However, we want a family of representations simultaneously of all B_n's and want compatibility of the trace under the move $B_n \ni \beta \mapsto \beta\sigma_n \in B_{n+1}$ In fact, Jones original construction of Jones polynomials used the idea of traces on representation of Braid groups.

8.3. **Temperly–Lieb algebras.** In this §we will follow a ahistorical construction of the Jones polynomials along the same lines as that of Jones but not as identically because he did not use Kauffmann bracket and his method is more complicated than what we want.

Temperly–Lieb algebras is a "planar" analogs of braids defined by

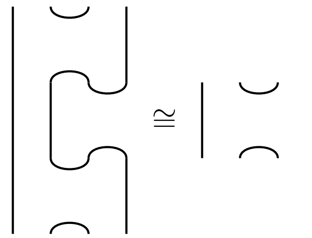
$$TL_n(\delta) \coloneqq \langle e_1, \ldots, e_{n-1} : e_i \leftrightarrow e_j \iff |i-j| < 1, \quad e_ie_{i+1}e_i = e_i, \quad e_i^2 = \delta e_i, \quad \delta \in \mathbb{C}\rangle.$$

Pictorially, e_i looks like

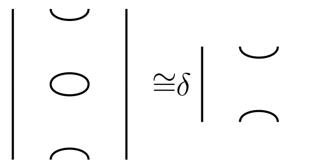


e_i

From the picture, the commutativity is clear. We can visualize $e_ie_{i+1}e_i = e_i$ as follows



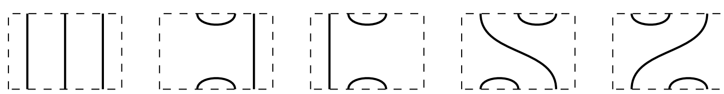
The third relation is saying that the value of an unknot is δ. Pictorially,



In particular, the Temperly–Lieb algebra has a pictorial description as follows:

$$TL_n(\delta) := \frac{\mathbb{C}\langle \text{isotopy classes of planar diagrams in box with } n \text{ inputs and output}\rangle}{\text{closed loop} = \delta}.$$

Example 8.10. The basis elements of TL_3 are $1, e_1, e_2, e_1e_2, e_2e_1$ visualized respectively as follows.

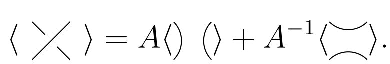


In general,

$$\dim TL_n = n^{th}\text{-Catalan}\# = \frac{1}{n+1}\binom{2n}{n}.$$

Remark 8.11. Jones had discovered the algebra and the pictures we discussed above. However he associated the algebra to Temperley and Lieb because they had written down a matrix representation of the algebra but they did not have pictures nor had any topological interpretation.

Recall that Kauffman bracket gives an invariant of framed links. To compute the Kauffman bracket, we smooth out K crossings into 2^K ways eventually getting unknots and use the relation



The above relation says that the Kauffman bracket converts the Braid group to Temperly–Lieb algebra (corresponding to the smoothing), i.e., there exists homomorphism

$$\psi : \mathbb{C}B_n \to TL_n(\delta = -A^2 - A^{-2})$$
$$\sigma_i \mapsto A\,\mathrm{id} + A^{-1}e_i.$$

Furthermore, there exists a trace function on $TL_n(\delta)$ corresponding to the taking the closure of β given by

$$B_n \ni x \mapsto \delta^{\#\text{loops got by closing up diagram of } x}.$$

Note that this trace map satisfies satisfies $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.

The upshot is that

$$\operatorname{tr} \psi(\beta) = \langle \hat{\beta} \rangle,$$

where $\hat{\beta}$ is viewed as a framed oriented link. We can correct both side of the above equation via the writhe if desired. Here, the write is defined as

$$w(\hat{\beta}) = \text{exponent of } \beta \in B_n.$$

By exponent, we mean that the sum of the powers of the σ_i's appearing in the decomposition of β into generators.

Definition 8.12. We could therefore say

$$(-A^3)^{-\exp(\beta)} \operatorname{tr}(\psi(\beta))$$

is the *Jones polynomials* of $\hat{\beta}$ (oriented link, unframed).

Remark 8.13. Jones went through the Hecke algebra (which is more complicated) in his actual paper. He did not use Kauffmann bracket. The Temperly–Lieb algebra structure is the one that he found in his operator algebra which he gradually managed to relate them to links.

8.4. **Representation of braid groups and Yang–Baxter operator.** Previously, we saw that

$$\mathbb{C}B_n \xrightarrow{\psi} TL_n(\delta = -A^2 - A^{-2}) \xrightarrow{\operatorname{tr}} \mathbb{C}$$

gives an interpretation of the Kauffmann bracket (and therefore of Jones polynomials) in terms of a trace map of "representation" of braid groups in terms of the Temperly–Leib algebra:

$$\operatorname{tr}(\psi(\beta)) = \langle \hat{\beta} \rangle.$$

Since Temperly–Leib algebra is not a vector space, ψ is not quite a representation. In this §, we will give an honest representation of the $TL_n(\delta)$ on a vector space as follows. Then using ψ, we get an honest representation of the braid groups.

Remark 8.14. In general, finding representation of braid groups (and taking trace) allows us to find invariants of braids and therefore links.

To define a represendation on a vector space V, we need to associate to each generators an element of $\operatorname{End}(V)$. For the rest of the section we set $V = \mathbb{C}\langle e_1, e_2 \rangle$ We will follow the procedure similar to what we did in $(0+1)$ TQFT to define a representation. The arrow of time is going upward.

First, $\cup$ goes to a map $\mathbb{C} \to V \otimes V$ which can be identified with an element

$$u = Ae_1 \otimes e_2 + A^{-1} e_2 \otimes e_1.$$

Similarly, $\cap$ goes to a map $n : V \otimes V \to \mathbb{C}$ defined by

$$e_1 \otimes e_2 \mapsto A$$
$$e_2 \otimes e_1 \mapsto -A^{-1}$$

$$e_1 \otimes e_1 \mapsto 0$$
$$e_2 \otimes e_2 \mapsto 0$$

Once we have these assigment, just like in $(0+1)$ TQFT, we can check that $\cong$ corresponds to

$$V \to V \otimes V \otimes V \to V$$
$$v \mapsto u \otimes v \overset{\mathrm{id} \otimes n}{\mapsto} v.$$

We have a similar correspondence for the following equivalence of diagrams $\cong$.

Finally, corresponds to $n \circ u$ which is equal to $-A^2 - A^{-2} = \delta$.

In any case, the above correspondence gives a representation

$$TL_n(\delta) \to \mathrm{End}(V^{\otimes n}),$$

after composition with ψ gives rise to the representation of $\mathbb{C}B_n$

$$\begin{aligned} \mathbb{C}B_n &\to TL_n(\delta) \to \mathrm{End}(V^{\otimes n}) \\ \sigma_i &\mapsto A\,\mathrm{id} + A^{-1}e_i \mapsto R_{i,i+1} = \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \otimes R \otimes \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \end{aligned}$$

where $R \in \mathrm{End}(V^{\otimes 2})$ is given by the matrix

$$R = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{pmatrix}.$$

Recall that

$$\sigma_i \sigma_{i+1} \sigma_1 = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

which under the representation implies that

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R). \tag{8.2}$$

For instance, when $i = 1$ the above equation becomes

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.$$

By (8.2), we say that "the R-matrix" satisfies the Yang Baxter equation. The Yang–Baxter equation is really just the Reidemeister III in disguise. An operator that satisfies the Yang–Baxter equation is called the Yang–Baxter operator.

Remark 8.15. The incarnation of braid groups in terms of the above respresentation does not help us compute the invariants easily. However, theoretically, it is an important way of thinking about $(0+1)$ TQFT applied to pictures of Temperly–Lieb algebra to get its action on some vector space.

9. Invariants of tangles and Ribbon category

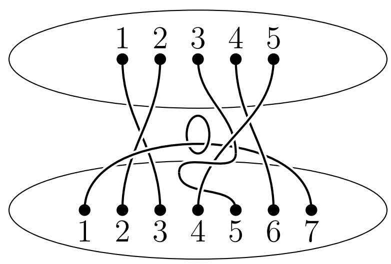
We categorify the idea of getting invariants of framed link via representation of the braid group. Namely, we will show that every time there is a functor from **Tang** to a Ribbon category (defined below), we get invariants of framed links just like how an $(n+1)$-TQFT gives rise to topological invariants of $n+1$ manifolds.

9.1. Tangles.

Definition 9.1. The category **Tang** of framed unoriented tangles consists of teh following data.

(1) Objects are natural numbers $\mathbb{N} \subset \mathbb{R}^2$.
(2) $\mathbf{Hom}(m, n) =$ set of (regular) isotopy classes of framed tangles in slab $\mathbb{R}^2 \times I$. Imlicitly, they are blackboard framed tangles.
(3) Composition is vertical stacking of tangles.

Pictorially, a tangle correponding to $\mathbf{Hom}(7, 5)$ could look like the picture below.



Remark 9.2. (1) $\mathbf{Hom}(0, 0)$ in **Tang** is the set of isotopy classes of framed links.
(2) **Tang** is a categorification of the braid groups. In particular, every time we have a representation of Braid groups, we could ask if it transfers to a representation of tangles.

9.2. **Tensor categories.** To define invariants of framed links, we study categorical properties of **Tang** and see how some topological properties of tangles transfer to algebraic properties. At the end, we will prove that **Tang** is the free Ribbon category on one object from which we will be able to define invariants of framed links.

Definition 9.3. A *monoidal category* $\mathcal{C}$ has a bifuntor $\otimes \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and associativity natural isomorphisms

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \arrow[r, bend left, "\otimes\cdot(1\times\otimes)", ""{name=U, below}] \arrow[r, bend right, "\otimes\cdot(\otimes\times\mathbb{1})"', ""{name=D}] & \mathcal{C} \arrow[from=D, to=U, Rightarrow, "\alpha"] \end{tikzcd},$$

i.e., there are maps $\{\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)\}_{A,B,C}$. These maps must satisfy the pentagon identity:

$$\begin{tikzcd}
 & (((A \otimes B) \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[ur, "\alpha^{-1}_{A \otimes B, C, D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A, B \otimes C, D}"] \\
A \otimes (B \otimes (C \otimes D)) \arrow[u, "\alpha^{-1}_{A,B,C \otimes D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "\mathrm{id}_A \otimes \alpha_{B,C,D}"]
\end{tikzcd},$$

so that the composition map is identity.

Remark 9.4. The pentagon identity implies that we can omit bracketings safely so that there exists a canonical object corresponding to tensor product of four objects.

Theorem 9.5 (MacLane's Coherence theorem)**.** *Given a pentagon, we have a canonical isomorphism between all bracketings of n-fold $\otimes$-products for all n. Also, there exists a unit object* $1 \in \mathcal{C}$ *with natural isomorphism* $\{A \otimes 1 \xrightarrow{\cong} A \xleftarrow{\cong} 1 \otimes A\}$ *satisfying a commutative diagram*

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "\alpha"] \arrow[dr] & & A \otimes (1 \otimes B) \arrow[dl] \\
 & A \otimes B &
\end{tikzcd}.$$

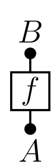
The commutativity structures are more interesting.

Definition 9.6. A *braiding on a monoidal category* $\mathcal{C}$ is a natural isomorphisms $\{\sigma_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A\}$ that satisfy the following hexagon commutative diagram:

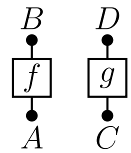
$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "\sigma_{A,B} \otimes \mathrm{id}_C"] \arrow[dl, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"] & \\
A \otimes (B \otimes C) \arrow[dr, "\mathrm{id}_A \otimes \sigma_{B,C}"'] & & & B \otimes (A \otimes C) \arrow[dl, "\sigma_{B, A \otimes C}"] \\
 & A \otimes (C \otimes B) & (A \otimes C) \otimes B \arrow[l, "\alpha_{A,C,B}"] &
\end{tikzcd}$$

9.2.1. *Graphical calculus.* We will pictorially formalize categorical structures.

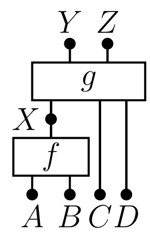
First a morphism $f : A \to B$ can be viewed as



Tensoring in $\mathcal{C}$ amounts to disjoint union, for instance $f : A \to B$ and $g : C \to D$ can be tensored to get and element $f \otimes g \in \mathbf{Hom}(A \otimes C, B \otimes D)$ which can be represented as follows.

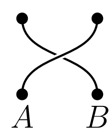


Composition of $f : A \otimes B \to X$ and $g : X \otimes C \otimes D \to Y \otimes Z$ is an element $f \circ g \in \mathbf{Hom}(A \otimes B \otimes C \otimes D, Y \otimes Z)$ that can be pictorially represented as follows.

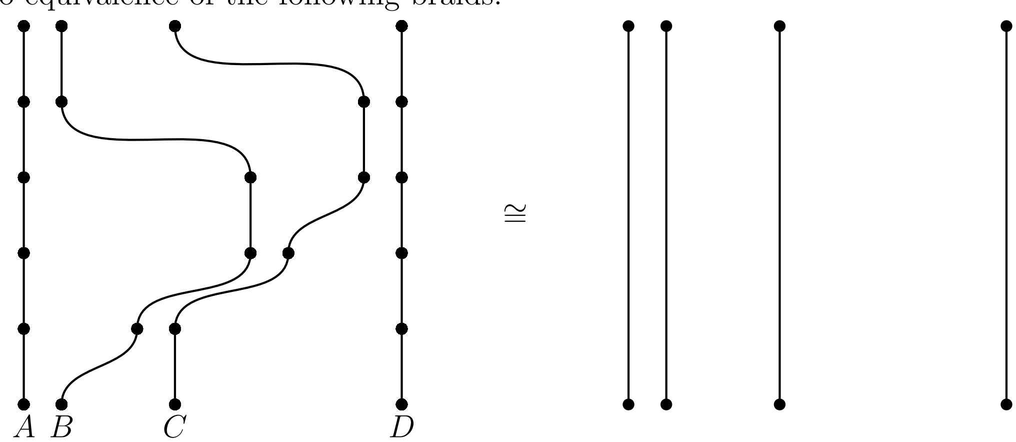


The above picture is a formalization of a planar version of what is called Penrose's tensor calculus.[4]

On the other hand, the braiding $\sigma_{A,B} : A \otimes B \to B \otimes A$ can be represented by the following crossing.



Then the commutativity of the pentagon diagram in a braided monoidal category corresponds to equivalence of the following braids.

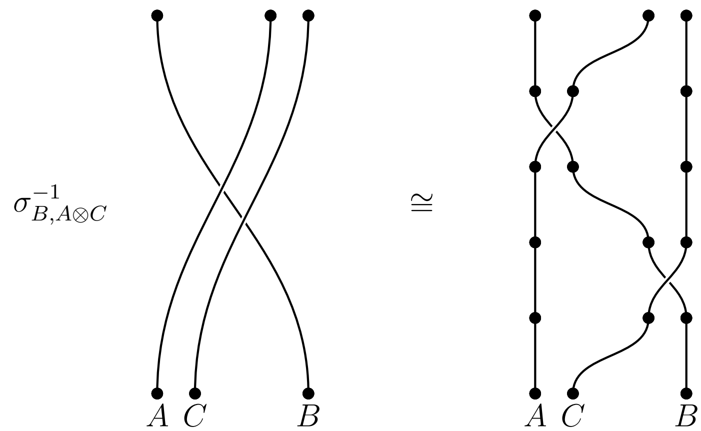


In the above picture, we the bottom corresponds to $((A \otimes B) \otimes C) \otimes D$ (top of the pentagon diagram) and going up corresponds to going clockwise in the diagram. We view bracketing

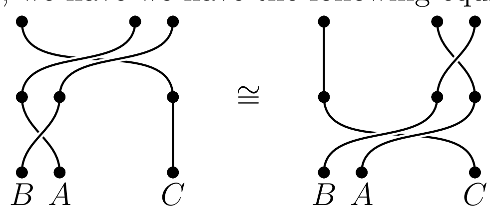
[4]Penrose represented tensor calculation in general relativity with Pensore diagrams. For instance, if we choose bases $\{a_n\}, \{b_o\}, \{c_k\}\{d_l\}, \{x_m\}, \{z_j\}, \{y_i\}$ of A, B, C, D, X, Y, Z respectively then above diagram represents $\sum_m g^{ij}_{klm} f^m_{no}$.

to represent closeness, i.e., in $((A \otimes B) \otimes C) \otimes D$, A and B are close and C is close to $A \otimes D$ than to D as show in the above picture.

Further, the commutativity of the hexagon diagram means the equivalence of the following braids:



Remark 9.7. Given the hexagon commutative diagram, Reidemeiter III relation will be satisfied automatically. In fact, we have we have the following equivalence of braids.



Using the identity $\sigma_{A,B} \circ \sigma^{-1}_{A,B} = \mathrm{id}$ we can similarly get the second Reidemeister move.

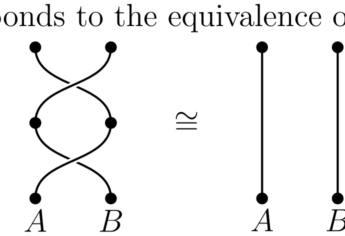
So far, we have pictorially, sketched out the proof of the following proposition.

Proposition 9.8. *The category of all braids is the free braided monoidal category on one object.*

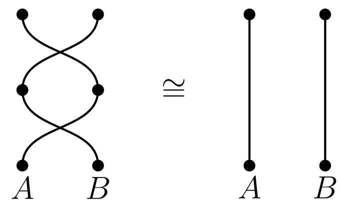
Remark 9.9.
(1) Here one object is a point and taking n tensor power amounts to taking n-points. There are only endomorphism between objects in the category of braids unlike in the category of tangles.
(2) Here, free means that there should be a functor from the category of all braids to any braided monoidal category. The functor can be characterized by where a point gets sent to. For instance, if a point goes to $C \in \mathcal{C}$ then n-points go to $C^{\otimes n}$.

Definition 9.10. A braiding is *symmetric* if $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{id}_{A\otimes B}$.

Pictorially, symmetric corresponds to the equivalence of the following braids.

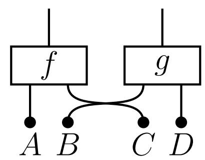


In the presence of symmetric braid, we can forget the type of crossings so the above equivalence becomes the following.



Remark 9.11. They symmetric structure is implicit in **Vect**, namely $A \otimes B \cong B \otimes A$ via a specific linear map that corresponding to two flips $a \otimes b \mapsto b \otimes a \mapsto a \otimes b$.

In terms of the Penrose's diagrams, we can therefore have diagrams like



where the crossing represents $b \otimes c \mapsto c \otimes b$.

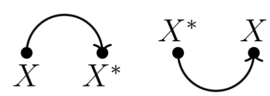
Example 9.12. The categories **Vect**, **Set**, **Group** are symmetric monoidal categories. Braiding does not come from "classical" mathematical structures. They come from quantum groups, which we will talk about soon.

A monoidal category can have *duality* as structure.

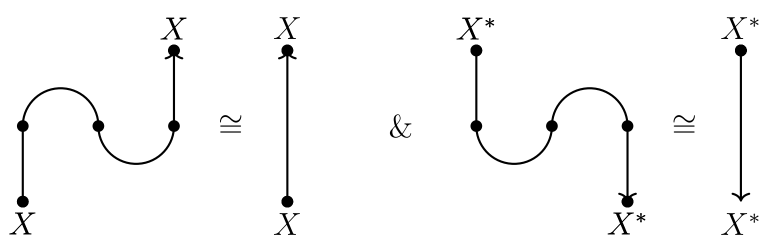
Definition 9.13. For any $X \in \mathcal{C}$, a *right dual*[5] is an object X^* with two morphisms

$$X \otimes X^* \xrightarrow{ev} 1,$$
$$1 \to X^* \otimes X,$$

satisfying the "S-bend" relations. Pictorially, we can represent the evaluation map and the coevaluation maps respectively as follows.



Then the S-bend relations are the equivalence of the following diagrams.

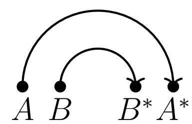


The straight morphism corresponds to the identity.

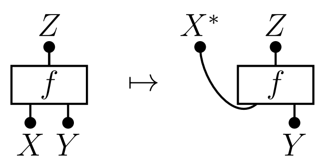
When duality is paired up with braiding in a monoidal structure, we get a lot more algebraic relations and structures.

[5]Our convention is different than the convention in [EGNO15].

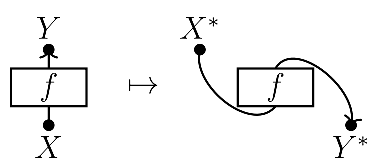
Remark 9.14. (1) A dual X^* is unique up to canonical isomorphism.
(2) X^* may not exist for all (or any!) objects in $\mathcal{C}$.
(3) If we have duals for all objects then we can make choices of X^*'s and then define a functor $X \mapsto X^*$.
(4) Similarly, we could define a left dual ${}^*X$ but there is not reason for them to coincide, in general. It is true though that ${}^*(X^*) = X = ({}^*X)^*$ (follows from the picture.)
(5) In a tensor category, we have $(A \otimes B)^* \cong B^* \otimes A^*$ which can be visualized pictorially as follows.



There are a parallel version of S-bend relations.
(6) Given a dual, we can start switching inputs and outputs on tensors. For example, there is a isomorphism between $\mathbf{Hom}(X \otimes Y, Z)$ and $\mathbf{Hom}(Y, X^* \otimes Z)$ via the correspondence of composing with a coevaluation map.



(7) Similarly, we can also dualize morphism by composing with coevaluation and evaluation as follows.



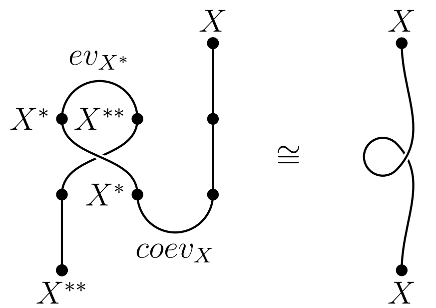
The map on the right correspond to the adjoint f^* of f.
(8) In general, X^{**} is not isomorphic to X. We must axiomatize (via *pivotal structure* below) the structure of double duality to make sure that $X^{**} \cong X$.

Definition 9.15. A *pivotal structure* on a category with duals is a natural isomorphism $\delta : X \to X^{**}$ satisfying

$$\begin{aligned} \delta_{X \otimes Y} &= \delta_X \otimes \delta_Y \\ \delta_1 &= 1 \\ \delta_{X^*} &= (\delta_X^*)^{-1}. \end{aligned}$$

A pivotal structure along with braiding and duality forces the double dual functor to be isomorphic to identity functor.

Just with a category with braiding as well as duals, the following diagram (on the left) corresponds to a morphism $X^{**} \xrightarrow{\theta} X$. Then composing with the pivotal structure gives rises to the isomorphism $X \to X^{**} \to X$ (corresponding to the right picture) and the following equivalence.

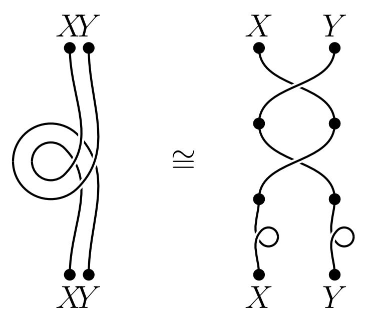


Here the isomorphism happens after composing the picture on the left with the pivotal

Further, the morphism θ satisfies the relation

$$\theta_{X \otimes Y} = \delta_{YX} \circ \sigma_{X,Y} \circ (\theta_X \otimes \theta_Y),$$

corresponding to the equivalence of the following pictures.



This above equivalence can be thought of as an equivalence of (blackboard) framed tangles.

Definition 9.16. A braided category with duals and compatible pivotal structure is called a *ribbon category.*

Remark 9.17. The word ribbon is used because as we saw above framed tangles (ribbons) are appropriate to denote the morphisms in the Ribbon category. framed means ribbons.

Actually, we have sketched out the proof of the following theorem.

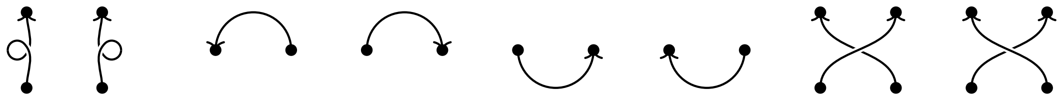
Theorem 9.18. *The category* **Tang** *of framed oriented tangles is the free ribbon category on one object.*

More generally, we have the following theorem.

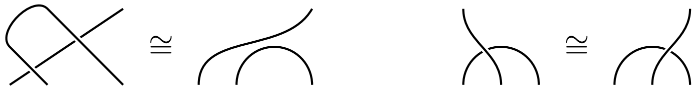
Theorem 9.19. *The category of labelled/colored/decorated framed oriented tangles, i.e., each component of a tangle is labelled by an element of some set S of colors, is the free Ribbon category on the set of objects S.*

It means that given any ribbon category $\mathcal{D}$ and a choice of objects $A, B, D \ldots, \in \mathcal{D}$ corresponding to elements in S, we get a functor F from framed oriented S-colored tangles to $\mathcal{D}$. In particular, for every tangle, we can assign the appropriate morphisms in the category $\mathcal{D}$. To do so, we need to understand use Morse theory to find generators and relations of colored tangles.

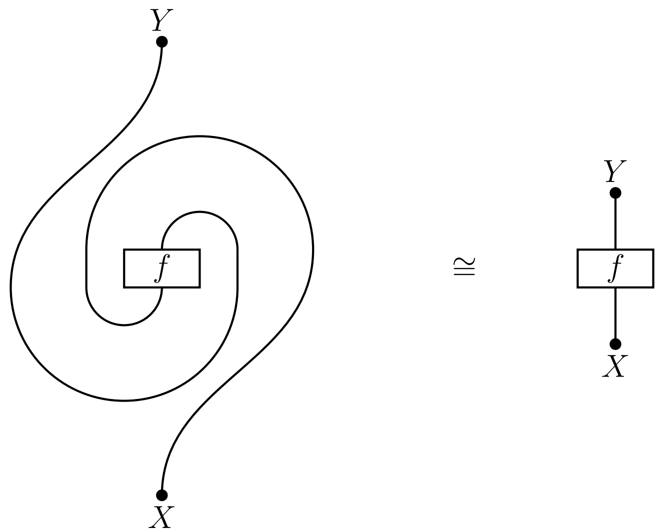
Just like what we did in (1+1)-TQFT, we can use Morse theory to prove that the generators of **Tang** are as follows considered with framings.



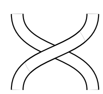
Some example relations for these generators are given below.



The "pivotal" structure is used to untangle. For instance, we have the following equivalence.

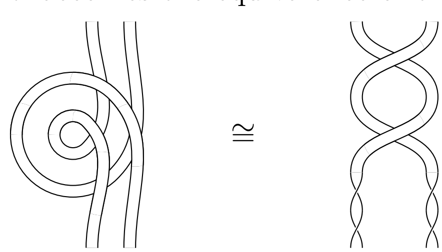


In the above diagrams, the framing is implicit. For instance, in blackboard framing (or as ribbons), we can view the positive crossing as the following picture.



Similarly, the following ribbon visualizes θ_X.

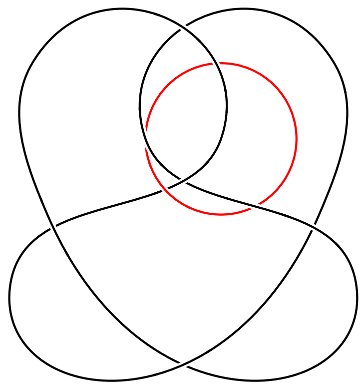
Then the axiom satisfied by θ becomes the equivalence of the following ribbons:



The upshot of the freeness of the category of colored tangles is the following theorem which is a recipe to produce link invariants.

Theorem 9.20. *Given a Ribbon category $\mathcal{C}$ and objects $\{X_i\} \in \mathcal{C}$, any framed link "decorated/labeled/colored" by $\{X_i\}$ will define an element on $\mathbf{Hom}_{\mathcal{C}}(1, 1) =$ "scalar."*

The point is that we can use Morse theory on a colored link to decompose it into generators that can be mapped to morphisms in $\mathcal{C}$ which can be composed together to get an element in $\mathbf{Hom}_{\mathcal{C}}(1,1) =$ "*scalar*." The scalar gives an invariant of the link.



10. Quantum groups and link invariants

Where do these Ribbon cateogries come from? Classical mathematics of vectors, sets and groups don't give rise to Ribbon categories. Ribbon categories come from quantum groups which as deformation of Hopf algebras. More precisely, the category of representations of quantum groups forms a Ribbon category. We will sketch out the heuristics of the previous statement.

10.1. Hopf algebra.

Definition 10.1. A *Hopf algebra* in a symmetric monoidal category $\mathcal{C}$ is an object $A \in \mathcal{C}$ with the following structure maps:

(1) Multiplication $\mu : A \otimes A \to A$.
(2) Comultiplication: $\Delta : A \to A \otimes A$.
(3) Antipode: $s : A \to A$.
(4) Unit $\iota : 1 \to A$.
(5) Counit $\epsilon : A \to 1$.

These maps satisfy the following axioms:

(1) A is an associative algebra with unit ι meaning μ is associative with unit ι, i.e.,

$$\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu) : A \otimes A \otimes A \to A.$$

Further, the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\cong"'] & A \\
A \otimes 1 \arrow[r, "\mathrm{id} \otimes \iota"'] & A \otimes A \arrow[u, "\mu"']
\end{tikzcd}$$

(2) It is a coassociative coalgebra with counit ϵ meaning that the two compositions in the following diagram agree.

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A \otimes A \arrow[r, bend left, "\mathrm{id} \otimes \Delta"] \arrow[r, bend right, "\Delta \otimes \mathrm{id}"'] & A \otimes A \otimes A \end{tikzcd}$$

Further, we have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & A \otimes A \arrow[d, "\epsilon \otimes \mathrm{id}"] \\ A \arrow[r, "\cong"'] & A \otimes 1 \end{tikzcd}$$

(3) The comultiplication and multiplication are compatible via a *bialgebra axioms*, which says "Δ is an algebra homomorphism and μ is a co-algebra morphism such that

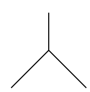
$$\Delta(\mu_A(a \otimes b)) = \mu_{A \otimes A}(\Delta(a) \otimes \Delta(b))$$

(4) The antipode satisfies the following:

$$\mu \circ (s \otimes \mathrm{id}) \circ \Delta = \iota \circ \epsilon.$$

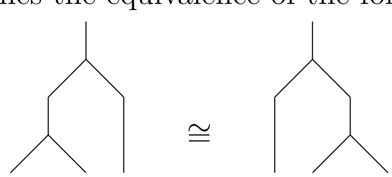
10.2. **Visualizing axioms of Hopf algebra.** We view time going upward and composition as stacking diagrams to visualize the axioms of Hopf algebra.

For instance, μ can be visualized as follows.

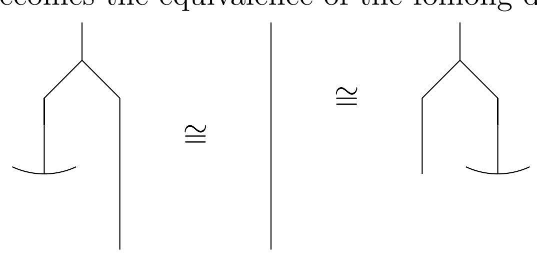


The unit becomes the following diagram.

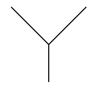
Then associativity axiom becomes the equivalence of the following diagrams.



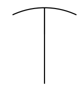
Further, the unit axiom becomes the equivalence of the folliong diagrams.



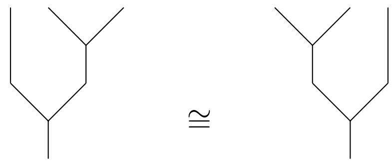
Similarly, we can visualize the comultiplication as follows.



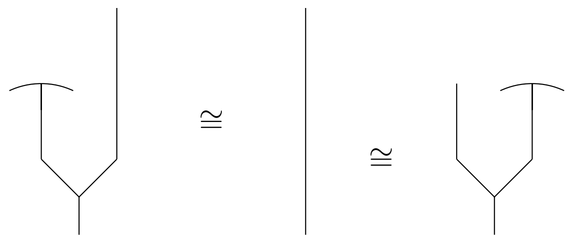
Further, the counit is visualized as follows.



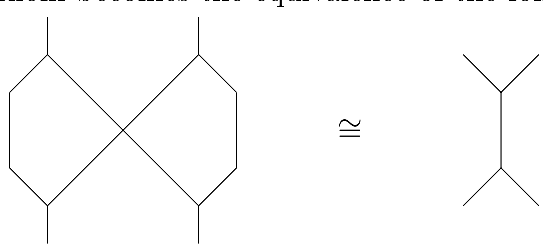
Then the cocommutativity axioms becomes the equivalence of the following diagrams.



Further, the co-unit axiom becomes the equivalence of the following diagrams.



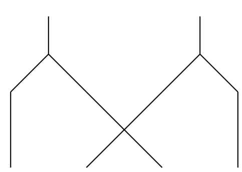
Finally, the bialgebra axiom becomes the equivalence of the following diagrams.



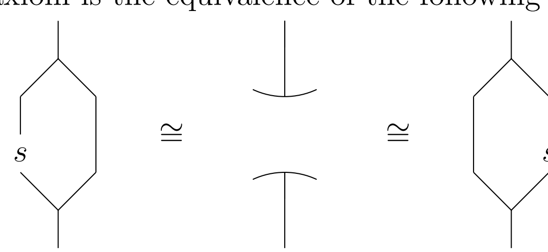
The reason for the existence of flip in the left-hand side is that multiplication in $A \otimes A$ has to be defined as

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2.$$

Therefore, $\mu_{A\otimes A}$ is visualized as follows.



Finally, the antipode axiom is the equivalence of the following diagrams.



10.3. **Examples of Hopf algebras.** Hopf algebras are generalization of ordinary associative algebra or generalization of groups. We can also view them as objects whose category of representation is a monoidal category.

Example 10.2. A Hopf algebra in **Set** is just a group. Recall that a group G has an associative multiplication $G \times G \to G$ with an identity element $1 \in G$ and inverse operation $g \mapsto g^{-1}$ such that $gg^{-1} = 1$.

Note that the counit is $\epsilon : G \to \{*\}$ since $\{*\}$ is the unit object in the category of sets. The counit axiom implies that

$$g \mapsto \Delta(g) \coloneqq (g_1, g_2) \mapsto g$$

which forces Δ to be the diagonal map.

The coproduct Δ is secretly used in defining inverse. In fact,

$$g \xrightarrow{\Delta} (g, g) \to (g^{-1}, g) \xrightarrow{\mu} 1$$

where s is the inverse.

Example 10.3. The group algebra $\mathbb{C}G$ of discrete groups forms a Hopf algebra once we set

$$\begin{aligned} \Delta(g) &= g \otimes g \\ s(g) &= g^{-1} \\ \epsilon(g) &= 1 \end{aligned}$$

for each $g \in G$ and for other $\mathbb{C}G$ the operations are extended linearly. For instance $\Delta(2g) = 2g \otimes g$ and NOT $4g \otimes g$.

Note that $\mathbb{C}G \otimes \mathbb{C}G \cong \mathbb{C}[G \otimes G]$. Then the antipode axiom says that

$$g \mapsto (g, g) \mapsto (g^{-1}, g) \to 1.$$

In general, the Hopf algebra is not commutative. However, it is cocommutative, i.e., the comultiplication is symmetric:

$$\sigma \circ \Delta = \Delta$$

where σ is the flip in the ambient tensor category.

Example 10.4. Similarly, $\mathcal{F}(G)$, $\mathbb{C}$ valued functions on G with pointwise multiplication defines a Hopf algebra. The multiplication is commutative. The identity is the constant function 1. The comultiplication is defined as

$$\begin{aligned} \Delta : \mathcal{F}(G) &\to \mathcal{F}(G \times G) \\ \Delta(f)(g_1, g_2) &= f(g_1 \cdot g_2). \end{aligned}$$

For the above to be well defined, we have to make sure that $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$, i.e., $\mathcal{F}(G) \otimes \mathcal{F}(G) = \mathcal{F}(G \times G)$. This is true when G is finite. When G is not finite, we would have to redefine tensor product in terms of completion. Note that the Hopf algebra has self-dual structures so having finiteness is natural. Finally, $\epsilon(f) = f(1_G)$.

Example 10.5. For a Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U_{\mathfrak{g}}$ is a Hopf algebra. Here the universal algebra is the freely generated by the elements with the relations of the Lie algebra. It is the associative algebra that envelopes the Lie algebra.

In general, the universal algebra is non-commutative. However, it is co-commutitative. In fact,

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

for $x \in \mathfrak{g} \subset U_{\mathfrak{g}}$ and it is extended multiplicatively. Therefore,

$$\sigma \circ \Delta = \Delta.$$

The antipode $s(x) = -x$ is the infinitesimal inverse. s extends to $U_{\mathfrak{g}}$. Finally, $\epsilon(x) = 0$.

Example 10.6 (A special case of this for abelian Lie algebra/superalgebra)**.** The symmetric algebra S^*V of a vector space V can be viewed as the universal enveloping algebra of V, where the Lie algebra structure is trivial, i.e. $[\cdot,\cdot] \equiv 0$.

The exterior algebra Λ^*V of a vector space V can be thought of as universal enveloping algebra of a superalgebra V with trivial Lie algebra structure.

Both of the special cases are commutative and co-commutative.

In general, we are instead in Hopf algebras that are neither commutative nor co-commutative.

10.4. Category of representation of Hopf algebra.

We will see that a Hopf algebra is the kind of algebra whose category of representation is a monoidal category with duals.

Fix an algebra A.

Definition 10.7. An *A-module* is $A \otimes M \overset{a_M}{\to} M$ satisfying obvious axioms:

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "1\otimes a_M"'] & A \otimes M \arrow[d, "a_M"] \\
A \otimes M \arrow[r, "a_M"'] & M
\end{tikzcd}$$

Let $\mathrm{Rep}(A)$ is the category of representations.

Remark 10.8. For a plain algebra A, there is no natural $\otimes$ of modules or dual or unit object. For instance, given A modules M and N, $M \otimes N$ is naturally $A \otimes A$ modules. In order to make them into A module, we need an algebra homomorphism $A \to A \otimes A$.

If A admits a counit ϵ, then unit object 1 in $\mathrm{Rep}(A)$ becomes an A-module via an algebra homomorphism $A \overset{\epsilon}{\to} 1$.

Remark 10.9. In the category of $\mathbb{C}$-vector space, an algebra does not necessarily act on the ground field $\mathbb{C}$. For instance, there is no homomorphism from A to $\mathrm{End}(\mathbb{C})$. However, if A has a counit, then $a \in A$ acts on $\lambda \in \mathbb{C}$ by mapping it to $\epsilon(a)\lambda$.

Antipode allows us to make duals. For example, in **Vect**, the dual of V is $V^* = \mathbf{Hom}(A, \mathbb{C})$. However, if V is a left module then V^* is a right module. To turn it into a left module, we need an anti-homomorphism from A to A. For instance, for $f \in V^*$, we can set $a \cdot f = f \circ s(a)$ using the anti-automorphism property of s to turn right action into the left action.

The antipodal axiom formula makes the natural pairing $V^* \otimes V \xrightarrow{ev} \mathbb{C}$ into a module map making the following diagram commute.

$$\begin{tikzcd} V^* \otimes V \arrow[r, "ev"] \arrow[d, "\text{acts by } a"'] & 1 \arrow[d, "\text{acts by } a"] \\ V^* \otimes V \arrow[r, "ev"'] & 1 \end{tikzcd}$$

Here, the action of a on $V^* \otimes V$ means that $\Delta(a)$ acts on $V^* \otimes V$. To understand the above commutative diagram, suppose $\Delta(a) = \sum a' \otimes a''$. Then at element level, we have the following commutative diagram.

$$\begin{tikzcd} f \otimes v \arrow[r, mapsto] \arrow[d, mapsto] & f(v) \arrow[d, mapsto] \\ \sum a' \cdot f \otimes a'' \cdot v \arrow[r] & \epsilon(a) f(v) \end{tikzcd}$$

To derive the commutativity, note that

$$\sum a' \cdot f \otimes a'' = \sum f \circ s(a') \otimes a'' v \overset{ev}{\mapsto} \sum f(s(a')a''v) = f(\mu(s \otimes \mathrm{id})\Delta(a) \cdot v) = \epsilon(a) f(v).$$

We have sketched out the proof of the following result.

Proposition 10.10. *Fix a Hopf algebra A. The representation category* $\mathrm{Rep}(A)$ *has a tensor product with a unit object and duals in the sense that we have made choice of dual object into A-module that is compatible with duality morphisms.*

10.5. **Quasi-triangular Hopf algebra.** To prove that $\mathrm{Rep}(A)$ is a Ribbon category, we are still missing a braiding in the representation category.

In general, in $\mathrm{Rep}(A)$ we will have $V \otimes W \not\cong W \otimes W$ because A is in general not co-commmutative. In fact, $V \otimes W$ and $W \otimes V$ are made into A-modules using the co-multiplication and flipped co-multiplication. Therefore, although they are isomorphic as vector spaces they might not be isomorphic as modules.

If however there exists an element $R \in A \otimes A$ which is invertible such that

$$(\sigma \circ \Delta)(\bullet) = R^{-1} \circ \Delta(\bullet) \circ R$$

inside $A \otimes A$ then we will get a natural intertwining operator

$$R_{V,W} : V \otimes W \to W \otimes V$$

given by the flip followed by action of R.

Furthermore, suppose that we have the following two axioms

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$$
$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12},$$

where R_{ij} means R sitting in the ith/j-th palces.[6] Then it follows that $R_{V,W} : V \otimes W \to W \otimes V$ satisfies the hexagon relation. It implies that they satisfy the Yang–Baxter equation which means that $\mathrm{Rep}(A)$ is a braided monoidal category.

[6]For examples if $R = \sum a' \otimes a''$. Then R_{32} means $\sum 1 \otimes a'' \otimes a'$.

Finally, to prove Ribbonness, we need twist structures $\theta_V : V \to V$ intertwining with the braiding in the correct way. For this we need the following data.

Let $u = \mu(s \otimes \mathrm{id})\sigma \circ R \in A$, where σ is the flip, we must be able to find v such that

$$v^2 = s(u)u$$

and some other stuff. This will lead to Ribbonness of $\mathrm{Rep}(A)$.

10.6. **Quantum groups.** Quantum groups arise as canonical one parameter family of deformations of universal enveloping algebras of Lie groups viewed as Hopf algebras.

Remark 10.11. (1) There is no real definition of what a quantum group is. People have different meanings for it.

(2) Classical Lie groups (compact simple Lie groups) $SU(n), SO(n), SP(n)$ and some exceptional ones can't be deformed. There exists a theorem to the effect that if we introduce a one parameter family $G(t)$ of groups then $G(0) \cong G(t)$.

For instance $SU(2)$ is the three sphere topological and the multiplication is of unit quaternions on it. The theorem says that there is no way to deform the multiplication formally using some parameter without getting a structure that is isomorphic to the original one.

This is a cohomological theorem. Deformation theory/cohomological argument or more. This is predated cohomology theory. One of the Hilbert's problems has to do with uniqueness of Lie group structure on topological groups. Killing and Cartan must have understood from the deformation theory from a different point of view than the cohomological point of view.

(3) Deformation is the same as quantization in some sense. One of the interpretation of quantization is deformation.

$U_{\mathfrak{g}} \rightsquigarrow U_{\mathfrak{g}}[[\hbar]]$ (formal power series). We will have Hopf algebra structure on it.

Quantization is finding deformed algebra with deformed structure. $C^\infty(M) \to C^\infty(M)[[\hbar]]$ given by $fg \mapsto fg + i\hbar\{f,g\} + \dots$.

$[\hat{f}, \hat{g}] = i\hbar\{f,g\}$.

10.7. **Special linear group.** As an example, we will consider the deformation of universal enveloping algebra of the special linear group $SL(2)$. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2) = \mathbb{C}\langle e, f, h\rangle$ is the span of three generators e, f, h with Lie brackets defined by

$$[h,e] = 2e, [h,f] = -2f, [e,f] = h.$$

The lie algebra $\mathfrak{sl}(2)$ is three dimensional.

The Universal Enveloping Algebra of $\mathfrak{sl}(2)$ as algebra can be defined as

$$U_{\mathfrak{g}} = \mathbb{C}\langle e, f, h\rangle/\text{commutation relations},$$

where $\mathbb{C}\langle e, f, h\rangle$ is multiplicatively generated.

Remark 10.12. (1) $U_{\mathfrak{sl}(2)}$ is an infinitesimal algebra whose structure is the enveloping algebra of the Lie algebra.

(2) It's category of representation is the same as the category of the representation of the lie algebra itself. If we want to turn the world of Lie algebra to the world of associative algebra then we have to turn to the universal enveloping algebra functor.

The $U_{\mathfrak{sl}(2)}$ is a Hopf algebra structure where all three of e, f, h are primitive generators i.e.,

$$\Delta(e) = e \otimes 1 + 1 \otimes e,$$
$$\Delta(f) = f \otimes 1 + 1 \otimes f,$$
$$\Delta(h) = e \otimes 1 + 1 \otimes h.$$

The antipode $s(e) = -e, s(f) = -f, s(h) = -h$ and the counit $\epsilon(e) = \epsilon(e) = \epsilon(f) = 0$.

Remark 10.13.
(1) Primitiveness is about trying to recover Lie algebra from the universal enveloping algebra.
(2) This is a non-commutative but cocommutative Hopf algebra. Note that $\sigma \circ \Delta = \Delta$.

There are several ways to deform $U_{\mathfrak{sl}(2)}$.

(1) We can form $U_{\mathfrak{sl}(2)} \otimes \mathbb{C}[[\hbar]]$ and deform the algebra and co-algebra structure using formal power series in the variable $\hbar$. Think of $\hbar$ being small, but we cannot specialize except for $\hbar = 0$, which should recover the classical $U_{\mathfrak{sl}(2)}$. We don't have to worry about the convergence.

    Morally, a deformation quantization procedure replaces some commutative structure with something that is non-commutative which is defined over formal power series in a variable $\hbar$. This approach is primordial.
(2) If we are more careful, we can actually define an algebra over $\mathbb{C}(q)$, rational functions in variable $q = e^{\hbar}$. This allows specialization to transcendental values of $q \in \mathbb{C}$, but not algebraic numbers like roots of unity (the denominator might vanish).
(3) It is possible by even more care with the denominators to define a version $U_{\mathfrak{sl}(2)}$ where q can be specialized to any non-zero complex number; however things behave very strangely when q is a root of unity.
(4) There is a fourth kind of quantum group which is special to the case $q \in \mathbb{C}$ is a root of unity. This case is more complicated.

10.7.1. *Formulae definign the universal enveloping algebra.* To define $U_{q\mathfrak{sl}(2)}$ (over $\mathbb{C}(q)$) we $q = e^{\hbar}$ secretly and also "$K = e^{H/2}$", i.e., it correspond to an exponential of the classical generator 'h' rather than to h itself. Then we define

$$U_{q\mathfrak{sl}(2)} := \mathbb{C}\langle E, F, K^{\pm 1}\rangle$$

with relations

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The first two relations deform $[h, e] = 2e$ and $[h, f] = -2f$) and the third deforms the relation $[e, f] = h$.

Remark 10.14. Its helpful to introduce quantum integers: $[n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$. The point being $q \to 1$, $[n] \to n \in \mathbb{N}$.

The $U_{q\mathfrak{sl}(2)}$ algebra structure is in fact isomorphic to the original one but not in a nice way. However, the coalgebra structure changes. In fact,

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$$s(K) = K^{-1}, \quad s(E) = -EK^{-1}, \quad s(F) = -KF,$$
$$\epsilon(K) = 1, \quad \epsilon(E) = 0, \quad \epsilon(F) = 0.$$

Note that Δ is not any longer co-commutative.

However, there is a "universal R-element" $R \in A \otimes A$ where $A = U_{q\mathfrak{sl}(2)}$ which makes the quasi-triangular structure, in particular, we have an almost form of co-commutativity:

$$\sigma \circ \Delta = R \cdot \Delta \cdot R^{-1}.$$

Explicitly, we have

$$R = q^{H \otimes H/4} \sum_{n \geq 0} \frac{q^{n(n-1)/4}}{[n]!} \left((q^{1/2} - q^{-1/2}) E \otimes F \right)^n. \tag{10.1}$$

Here, $[n]! = [n][n-1][n-1] \dots [1]$.

Remark 10.15. Strictly speaking R lies in a completion of the algebra because it is an infinite sum which does not make sense if we think of the algebra as defined over $\mathbb{C}(q)$. It makes sense if we write $q = e^{\hbar}$. In any case, we can pretend its okay because it acts by a finite sum in any finite dimensional representations of the algebra A.

The Ribbon structure is completed by elements u and v of A defined oby

$$u = q^{-H^2/4} \sum q^{3n(n-1)/4} \frac{(q^{1/2} - q^{-1/2})^n}{[n]!} F^n K^{-n} E^n$$
$$v = K^{-1} u.$$

Remark 10.16. The denominators $[n]!$ give problem when we try to specialize to roots of unity. Because $[n]$ vanish if q is a root of unity.

The upshot is that $U_{q\mathfrak{sl}}(2)$ is a Ribbon Hopf algebra.

10.8. **Representations of quantum groups and link invariants.** We will study the representation of $U_{q\mathfrak{sl}}(2)$.

A fundamental fact is that the category of representation of $U_{q\mathfrak{g}}$ (where q isn't a root of unity) is equivalent to the usual category of representation of $U_{\mathfrak{g}}$ (but not as tensor category) which is the same as the category of representation of Lie algebra $\mathfrak{g}$. This allows us to study the representation of $U_{q\mathfrak{g}}$ by studying the representation of $\mathfrak{g}$.

10.8.1. *Fundamental representation.* The standard representation $\rho : \mathfrak{sl}(2) \to \mathrm{End}(\mathbb{C}^2)$ is the two dimensional representation on $V = \mathbb{C}^2$ given by

$$\rho(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the induced representation of $U_{q\mathfrak{sl}(2)}$, the action of E and F are the same as that of e and f respectively but K acts by $\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$.

Recall that the R matrix is the endomorphism of $(V \otimes V)$ associated to the crossing in the category of framed oriented tangles. With the help of R-matrix we know the Skein relation.

In any case R matrix in $\mathrm{End}(V \otimes V)$ for this representation is just

$$\rho(R) = \rho(e^{H \otimes H/4})(1 \otimes 1 + (q^{1/2} - q^{-1/2})\rho(E) \otimes \rho(F)).$$

Note that e and f are nilpotent. In fact, $\rho(e)^2 = \rho(f)^2 = 0$. Therefore, the higher term appearing in (10.1) vanish. Thus

$$R = \begin{pmatrix} q^{1/4} & & & \\ & q^{-1/4} & & \\ & & q^{1/4} & \\ & & & q^{-1/4} \end{pmatrix} \left(\mathrm{id}_{V \otimes V} + (q^{1/2} - q^{-1/2}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

From this and the Yang–Baxter equation for the R matrix, we can get the skein relation

$$q^{1/4}R - q^{-1/4}R^{-1} = (q^{1/2} - q^{-1/2})\,\mathrm{id}\,.$$

In other words,

$q^{1/4}$  $- q^{-1/4}$  $= (q^{1/2} - q^{-1/2})\uparrow\uparrow.$

This is the Skein relation for "framed Jones polynomials." It is really Kauffman bracket in disguise which is Jones polynomial without the correction for framing. To add a correction for framing, we can calculate the twist element as

 $= q^{-3/2}\uparrow.$

We could renormalize the Kauffman bracket using writhe we will can keep working with framed links.

Similarly, for $\mathfrak{sl}(N)$, the calculation will give us the Skein relation

$q^{1/2N}$  $- q^{1/2N}$  $= (q^{1/2} - q^{-1/2})\uparrow\uparrow,$

such that

 $= q^{N^2-1/N}\uparrow.$

This gives us a specialization of HOMFlY polynomial, if we were to correct via writhe.

Conversely, knowing all these quantum $\mathfrak{sl}(N)$ invariants of a link determines the HOMFLY polynomials, i.e., one two-variable polynomial (HOMFLY) is same as a family parameterized by N of one variable polynomials (the quantum $\mathfrak{sl}(N)$ invariants with the fundamental representation $\mathbb{C}^N$).

Remark 10.17. If we use $\mathfrak{so}(N), \mathfrak{sp}(N)$, we recover Kauffman polynomials. Leaving aside the exceptional group, for the other Lie algebras the quantum invariants in their fundamental representation recover HOMFLY polynomials.

10.8.2. *Higher irreducible representations.* We discuss invariants coming higher irreducible representations of $\mathfrak{sl}(2)$. In the representation ring of $\mathfrak{sl}(2)$, we denote V_1 to be the fundamental representation. Then any irreducible representation can be written as

$$V_n = S^n V_1,$$

where $S^n V_1$ is the n-the symmetric power of the fundamental representation. V_n has dimension $n+1$. The index n is by weight and not by dimension as is common is some books.

In the representation ring it turns out that

$$V_1 \otimes V_1 = V_0 \oplus V_2.$$

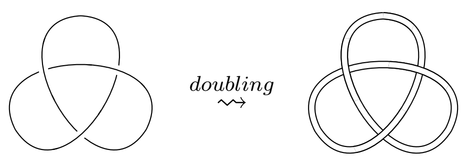
In terms of the knot invariants, this means that the quantum invariant $Q_{V_1 \otimes V_1}$ and $Q_{V_0 \otimes V_2}$ of a knot K associated to the above representations above satisfy

$$Q_{V_1 \otimes V_1}(K) = Q_{V_0}(K) + Q_{V_2}(K).$$

It means that labeling the knot with the representation $V_1 \otimes V_1$ will be the same as labeling the knot with representation V_0 plus the representation V_2. Note that $Q_{V_1 \otimes V_1}(K)$ in the calculus of Ribbon tensor category is associated to disjoint copies of knots that are *double parallel.* Further, since V_0 is the trivial representation $Q_{V_0}(K) = 1$ is the invariant of K labeled by trivial representation. In particular,

$$Q_{V_2}(K) = Q_{V_1}(K^{(2)}) - 1$$

where K^2 means K doubly parallel. Here, doubly parallel means that it is a framed knot put multiple of two knots using the framing to decide how they twist around one another. If we use blackboard framing then we just have a thickened knot into two parallel strings. See the picture below for a double parallel trefoil.



In general, we have the following result giving us a recipe to define a family of knot invariants by taking parallel copies of it.

Proposition 10.18. *The invariants of a knot K labeled by higher irreducible representations of $\mathfrak{sl}(2)$ are linear combinations of the invariants of parallels of K, all labeled by the fundamental representations.*

11. Physical origin of knot polynomials

We will discuss Witten's explanation and generalization of knot polynomials [Wit89].

Let G be a simply connected compact Lie group. Let M^3 be a closed oriented 3-manifold containing a framed link L with link components $\{L_i\}$ each of which is labeled by representations of G. Let $P \to M$ be a principal G-bundle on M. The simply connectedness of M implies that the G-bundle is necessarily trivial $M \times G$. Suppose $\mathcal{A}/G$ is the space of connection of P, up to gauge equivalence. Fix $k \in \mathbb{N}$. We define the path integral:

$$Z_{G,k}(M, L, \{V_i\}) \text{“} := \text{”} \int_{\mathcal{A}/G} e^{2\pi i k \mathrm{CS}(A)} \prod_{\{L_i\}} \mathrm{tr}_{V_i}(\mathrm{hol}_{L_i}(A))\, DA, \tag{11.1}$$

where $\mathrm{CS}(A)$ is the Chern–Simons invariant of A, $\mathrm{hol}_{L_i}(A)$ is the holonomy of the connection around the i-th component and tr_{V_i} is the trace in the representation V_i.

Remark 11.1.
(1) As of now, we don't have a rigorous mathematical interpretation of (11.1). The Lebesgue measure DA does not exist because $\mathcal{A}/G$ is an infinite dimensional space. In fact, it is one of the millennium problem posed by Clay Institute to formulate a theory where the above makes sense.
(2) Physicist assume that the integral makes sense and make formal inference from the integral, which works remarkably.
(3) The current state of art of the path integral is like that of infinite series in 19th century. There was no analytic definition of an infinite series. Mathematician were using common sense to compute values of certain infinite series.
(4) Z if we assume to be well-defined is a complex number, whose values are related to link polynomials.

Claim 11.1.1. *Z is a topological invariant of $(M, L, \{V_i\})$.*

Remark 11.2.
(1) The special case is when L is empty. These are topological invariants of 3-manifolds.
(2) Most QFT that physicist deal with are metric dependent.

The physical language is that $[\prod_{\{L_i\}} \mathrm{tr}_{V_i}(\mathrm{hol}_{L_i}(A))]$ is an observable and

$$\frac{Z_{G,k}(S^3, L, \{V_i\})}{Z_{G,k}(S^3)}$$

is the expectation value of the observable. It turns out that

$$Z_{G,k}(S^3, L, \{V_i\})/Z_{G,k}(S^3) = Q_{\mathfrak{g}}(L, \{V_i\})|_{q=e^{2\pi i/k+h}},$$

where h is the dual Coxeter number of G. For example, $h = 2$ in the case of $SU(2)$ or $SL(2)$.

Remark 11.3.
(1) It is not clear how framing of link components is actually used in the formula. The holonomy does not need framing.
(2) To be correct, we must also add some extra *framing-type* structure on M itself ("anomaly"). As it stands the formula is well-defined only up to powers of some (other) root of unity.
(3) We have very peculiar "shift" by h. In Turaev's approach there is no h appearing. We get positivity for this principal value. In general, we don't get positivity.
(4) If we change the orientation of the manifold then Z gets conjugated because the $CS(A)$ gets changed. What happens to the observable under change of orientation? The shift seems to be somewhat inconsistent with the change in orientation.

(5) It turns out that the values for 3-manifolds $\neq S^3$ do NOT come from specialization of polynomials in this way. k is known as level. We can think about k as reciprocal of the planck's constant. $k \sim 1/\hbar$.

Question premature: From the quantum group point of view, if we build these invariants using QG thinking way, we will come up with. There is Galois symmetry present. Choose root of unity of fixed order. We get different values for different choice.

For S^3 we don't need to choose a root of unity.

11.1. **Perturbation theory for path integral without the links.** Before analyzing the path integral (11.1) with links, we will focus on that just involves the Chern–Simons functional:

$$Z_{SU(2),k} \coloneqq \int_{\mathcal{A}/G} e^{2\pi i k CS(A)} DA, \quad k \in \mathbb{N}.$$

Note that the integrand makes sense because $CS(A) \in \mathbb{R}/\mathbb{Z}$.

A physicist analyzes the above path integral using perturbation theory by taking $k \to \infty$. Taking the semi-classical limit $\frac{1}{k} \sim \hbar \to 0$ amounts to dequantization to go to "classical physics." The main idea of the perturbation theory is to use the *stationary phase formula.* In the case of smooth functions $f : \mathbb{R} \to \mathbb{R}$, as $k \to \infty$, we get an asymptotic expansion in $\frac{1}{k}$

$$\int_{-\infty}^{\infty} e^{ikf(x)} dx \sim \sum_{p: f'(p)=0} \int_{-\infty}^{\infty} e^{ik(f(p)+\frac{1}{2}(x-p)^2 f''(p)+\dots)}\, dx.$$

The idea is that near points x that are not stationary, the phase $e^{ikf(x)}$ will whirl around the origin rapidly as $k \to \infty$ and will cancel out. The only contribution come from the critical points. Note that the quadratic term in the exponent is like Gaussian. There are techniques to deal with such Gaussian integrals based on

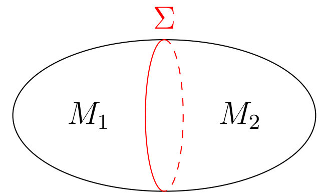
$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}.$$

The main insight is that $Z_{SU(2),k}$ becomes an asymptotic sum over critical points of CS. Recall that the critical points of CS are flat connections on M.

$$Z_{SU(2),k} \sim \sum_{\text{flat connections on } M} (\text{local contributions}).$$

Remark 11.4. (1) The QFT technique of analyzing the perturbative expansion, which basically amounts to computing integrals of "Gaussians" akin to that in finite dimension, gives a power series of topological invariants of manifolds.

(2) There is a completely different part of the story where one defines the individual coefficients in the power series using the configuration space integrals on 3-manifolds. They generalize the Gauss linking integral for two component knots in $\mathbb{R}^3$.



The QFT logic says that if we have a 3-manifold divided along a closed oriented surface Σ (see picture above), then we get a propagation idea

$$\begin{aligned}\int_{\mathcal{A}/G(M)} e^{2\pi ikCS(A)}\, DA \\ = \int_{A_\Sigma \in \mathcal{A}/G(\Sigma)} \left(\int_{\mathcal{A}/G(M_1, A_\Sigma)} e^{2\pi ikCS(A_1)}\, DA_1\right)\left(\int_{\mathcal{A}/G(M_2, A_\Sigma)} e^{2ikCS(A_2)}\, DA_2\right) DA_\Sigma.\end{aligned} \tag{11.2}$$

Here, A_i is connection on M_i extending A_Σ. The above equality can be thought of as running over the space of connections over M by first running over connections over Σ and for each connection over Σ we run over all the connection over M_i that extend the connection on Σ.

The key observation used in the above decomposition (11.2) is that

$$CS(A) = CS(A_1) + CS(A_2).$$

Remark 11.5. In physics literature, $\mathrm{tr}(\alpha \wedge d\alpha + 2/3\alpha \wedge \alpha \wedge \alpha)$ is called the *Lagrangian density* and its (Chern–Simon) *action* is $\int_M$ Lagrangian density.

The decomposition (11.2) is a justification for the functoriality of the quantum field theory. In fact, the path integral on the right of (11.2) is analogous to the pairing structure $\langle M_1 \mid M_2\rangle$ in "Hilbert space" $\mathcal{H}(\Sigma)$ that a TQFT assigns. However, we have the following question.

Question 11.6. What is the vector space associated to Σ in which this pairing takes place?

A naive thing would be to imagine the Hilbert space $\mathcal{H}(\Sigma) = L^2(\mathcal{A}/G(\Sigma))$ and

$$|M_2\rangle = \left(\int_{\mathcal{A}/G(M_2, A_\Sigma)} e^{2\pi iCS(A_2)}\, DA_2\right)(A_\Sigma).$$

And formally the path integral in (11.2) is $\langle M_1 \mid M_2\rangle$.

11.2. **Geometric quantization.** We are expecting to get a TQFT so we are looking for finite dimensional vector spaces. The philosophy of quantization says the following: we should

> Look at moduli space of "classical solutions" (critical points $\delta CS(A) = 0$ of the action functional) in vicinity $\Sigma \times \mathbb{R}$ of Σ and geometrically quantize it.

Recall that the critical points are flat connections on Σ. The space of flat connections can be identified with

$$\mathbb{M}(\Sigma) \coloneqq \mathbf{Hom}(\pi_1\Sigma, G)/G^{conj}. \tag{11.3}$$

In fact, picking a base point on Σ, flat connections are determined by their holonomy functions (homomorphisms $\pi_1\Sigma \to G$) considered up to overall conjugation.

It turns out that $\mathbb{M}(\Sigma)$ is a finite dimensional space. In fact, if Σ has genus g then

$$\pi_1\Sigma = \langle a_1, \dots, a_g, b_1, \dots, b_g : \prod_{i=1}^{g}[a_i, b_i] = 1\rangle.$$

Thus, $\mathbb{M}$ is almost a manifold (or more precisely an real algebraic variety), but there are singularities coming from quotient by the group action via conjugation.[7] It turns out that $\mathbb{M}$ is a symplectic manifold and therefore can be quantized using "geometric quantization."

To do so, we fix a hermitian holomorphic line bundle $\mathcal{L} \to \mathbb{M}(\Sigma)$ with respect to some Kähler structure on $\mathbb{M}(\Sigma)$ with

$$c_1(\mathcal{L}) = \text{symplectic form on } \mathbb{M}(\Sigma).$$

Here, $c_1(\mathcal{L}) = -\frac{1}{2\pi i}\operatorname{tr} F = -\frac{1}{2\pi i}F$ is the first Chern class, where the second equality follows because we are looking at line bundles.

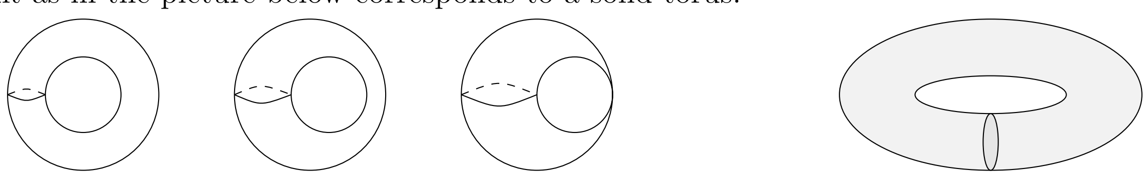
Consider the vector spaces

$$H^0(\mathbb{M}(\Sigma), \mathcal{L}^{\otimes k}) = \{\text{holomoprhic sections of the } k\text{-th power of the bundle } \mathcal{L}\}.$$

Remark 11.7. The construction of the vector space can be done using algebraic geometry. The result of this is the space of conformal blocks in Conformal Field Theory. People have a well understanding of their dimension.

The space $H^0(\mathbb{M}(\Sigma), \mathcal{L}^{\otimes k})$ is the required finite dimensional vector space that the TQFT assigns.

Remark 11.8. If we were to formalize the above assignment of vector spaces, we need to be able to create from a 3 manifold M_1 bounded by a Riemann surface Σ some vectors $|M_1\rangle$. However, at the time of this writing, people don't understand how to do that. The best attempt is to think of a three manifold using Morse theory as a family of Riemann surfaces that degenerate. If a curve on a Riemann surface degenerates to a node or split, that corresponds to a handle attachment. For instance, a family of torus acquiring nodal point as in the picture below corresponds to a solid torus.



As we move from one Riemann surface to another, there is a parallel transport of a connection of a bundle. We can compute the holonomy of the parallel transport and see what happens at the node. This is not entirely worked out differential geometric way. If it is worked out it is done at a combinatorial level via Moore–Seiberg equations. MS equations check at the combinatorial level that the combinatorics of the degenerating Riemann surface match the combinatorics of the three manifold. 

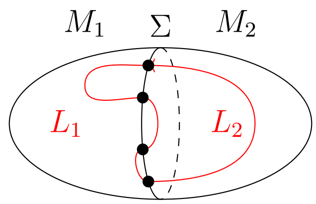
AG people can keep track of the behavior of the conform blocks as the Riemann surfaces degenerate.

[7]Groups with non-trivial stabilizer correspond to singularities after taking quotient.

If we just want to build a TQFT, we don't have to care about all this stuff. Ultimately, no one knows how to build a TQFT just using algebraic geometry or just using differential geometry. There is some combinatorics involed in it. We will soon see a combinatorial recipe based on knot polynomials to construct TQFT for three manifolds.

11.3. **Remark about Witten's paper.** The key point of Witten's paper [Wit89] is that there should be a TQFT associated with finite dimensional vector spaces as expected. The same conclusion holds when links are incorporated.

In fact, suppose (M, L) is built from (M_1, L_1) and (M_2, L_2) glued along the boundary Σ with marked points, see the picture below. The link is associated to a representation V of the group G.

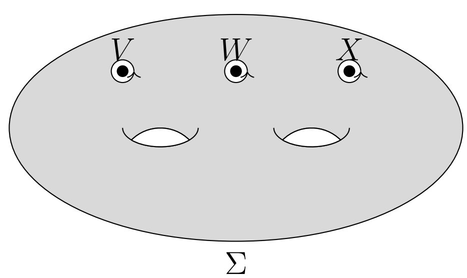


Then the pairing

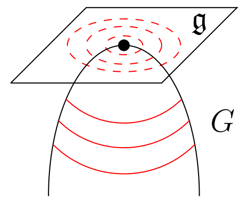
$$\langle M_1, L_1 \mid M_2, L_2 \rangle = Z_{G,k}(M, L)$$

would give an invariant of M^3 decorated with a link L. Here, $|M_i, L_i\rangle$ are vectors in the space $\mathcal{H}(\Sigma$ with marked points). The idea is that each link component inside M^3 will be associated to a representation of the quantum group G. From the representation, we should get the invariant $Z_{G,k}(M, L)$. Another way to get the invariant is to think of the link in terms of the marked points on Σ labeled by representation of G. This should also give the same link invariant.

We can think of marked Riemann surfaces as punctured Riemann surface. Each puncture is labeled by the representation V, W, X etc of a quantum group G. Then we look at the moduli space of flat connections on the punctured surface Σ with specified conjugacy classes of holonomy around each punctures, see the figure below. The moduli space is still a finite dimensional symplectic manifold.

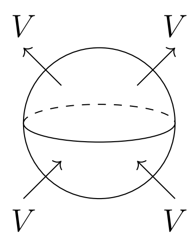


Remark 11.9. The correspondence between representations and conjugacy classes is out of the scope of these notes but it could be thought of as a quantized version of Borel–Weil–Bott theorem: In the classical representation theory the irreducible representations of a compact group can be obtained by quantizing the adjoint orbits. If we take the dual Lie algebra $\mathfrak{g}^*$ with the action of G to get orbits that are naturally symplectic. The orbits with integral symplectic form can be quantized using geometric quantization. The holomorpic sections of line bundles over those integral orbits gives the irreducible representation of G. In particular, there is a correspondence between integral coadjoint orbits and irreducible representations of G. The adjoint orbits on $\mathfrak{g}$ (dashed red lines) correspond to the conjugacy class (solid red lines) in the group G.

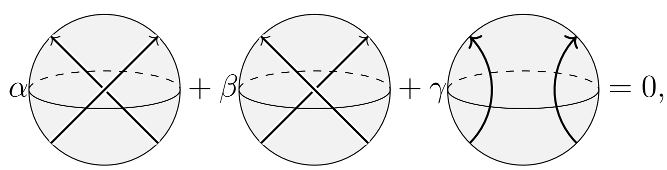


The quantized version of the above correspondence. Instead of integral coadjoint orbits and the irreducible representation of a classical Lie group, there is a correspondence between certain integral conjugacy classes with respect to a level and the representation of the quantum group.

The punchline of [Wit89] is that for $SU(2)$, the dimension of the vector space associated to S^3 with 4 points labeled by the fundamental representation V of $SU(N)$ is 2.



This means that any three vectors in the associated vector space for example those associated to B^3 containing tangles must be linearly dependent. Thus, there must exists a Skein relation of the form:



for some α, β, γ.

Witten is able to determine α, β, γ for $SU(N)$ using calculations in Conformal Field Theory which was known before Witten. These α, β, γ turn out to be the ones appearing in the Skein relation for Jones polynomials.

He also determines the space for a torus with no punctures. The dimension is $k + 1$. Further, he knows the action of the mapping class group $SL(2, \mathbb{Z})$ on the vector space using ideas from CFT. From this he can do some calculations which is out of the scope of these notes.

If we know a TQFT exists, we can pin down the properties by combinatorial fiddling around. That's the route we take. After Witten's paper, mathematicians were able to completely describe the TQFT by alternative combinatorial method. Reshetikin and Turaev used quantum groups representations. Lickorish did it using just the Kauffmann bracket for $SU(2)$.

Question 11.10 (Personal question)**.** How is geometric quantization that Witten alludes to related to the symplectic reduction of the space of connections to the space of flat connections?

12. A crash course on Chern–Weil theory

12.1. **Connections.** Consider a $\mathbb{C}^k$-fiber bundle

$$\begin{array}{ccc} \mathbb{C}^k & \longrightarrow & E \\ & & \downarrow \\ & & M. \end{array}$$

Locally, $E \to M$ can be written as the projection map $\pi_U : \mathbb{C}^k \times U \to U$. We denote the fibers of E at e to be E_e.

Let $\{s_1, \ldots, s_k\}$ be a basis of sections of E. Then we can write an arbitrary section s as a vector valued function:

$$s = \sum_{i=1}^{k} u_i s_i$$

where $u_i \in C^\infty(U)$. We can write $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}$ to pack coordinates of s.

Question 12.1. How do we differentiate sections along a vector field?

Remark 12.2. It does NOT make sense to just differentiate $\underline{u}$ along a vector field because the differentiation would depend on local coordinates.

To define derivative of section, we need a structure caleld covariant derivative.

Definition 12.3. A *covariant derivative* ∇ is a map

$$\nabla : C^\infty(E) \times \mathrm{Vect}(M) \to C^\infty(E),$$
$$(s, X) \mapsto \nabla_X$$

where $\mathrm{Vect}(M)$ is the space of vector fields on M, such that following holds

(1) $C^\infty(M)$ linearity in X: For any $f \in C^\infty(M)$ and $s \in C^\infty(E)$,

$$\nabla_{fX} s = f \nabla_X s.$$

(2) C^∞ derivation in s (or Leibniz rule): For any $f \in C^\infty(M)$ and $s \in C^\infty(E)$,

$$\nabla_X(fs) = (df)(X)s + f\nabla_X s.$$

Remark 12.4. Alternatively, we can view the covariant derivative as

$$\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$$

satisfying the Leibniz rule, i.e., for any $f \in C^\infty(M)$ and $s \in C^\infty(E)$,

$$\nabla(fs) = df \wedge s + f\nabla s.$$

Here, $\Omega^*(M;E) \coloneqq \Gamma(\Lambda^* T^* M \otimes E)$ is the section of the exterior powers of cotangent bundles valued in E.

In this interpretation, $\nabla_X s$ means that we evaluate the one form ∇s on vector X. Locally, with respect to a basis of sections $\{s_i\}$ over $U \subset M$ and local coordinates $\{x_i\}$ on U we can write ∇_X

$$\nabla_{X_i} s_j = \sum A_{ij}^k s_k,$$

where $A \in \Omega^1(U, \operatorname{End} E)$ and $X_i = \partial_{x_i}$. We think of A as a matrix valued 1-form or as a matrix of 1-forms.

For general section $s = \sum u_i s_i$ for $u_i \in C^\infty(U)$ we can think of s as $\underline{u}$ (vector valued function). Then

$$\nabla \underline{u} = d\underline{u} + A\underline{u}.$$

With respect to a different basis $t_j = \sum g_{ji} s_i$, $g_{ij} : U \to GL(k, \mathbb{C})$, we get a gauge transformation formula,

$$A \mapsto g^{-1} dg + g^{-1} A g.$$

Globally, any two connections ∇ and ∇' differ by a matrix valued one form:

$$\nabla' - \nabla = a \in \Omega^1(M, \operatorname{End} E).$$

In particular, $\nabla' - \nabla$ is an algebraic (0-th order) differential operator. Therefore, a choice of a local coordinates gives us a local choice of trivial connection ∇_0 relative to which we can write other connection in terms of the associated one-form.

12.2. **Parallel transport.**

Definition 12.5. A section s is *covariantly constant* along a path γ in M is

$$\frac{D}{dt} s = \nabla_{\dot\gamma(t)} s = 0 \text{ along } \gamma. \tag{12.1}$$

In local coordinates, (12.1) means that

$$\frac{d\underline{u}(\gamma(t))}{dt} + A(\dot\gamma(t))\underline{u} = 0. \tag{12.2}$$

Remark 12.6. Note that (12.2) is a system of first order ODEs. It admits a series of solutions to this equation $\underline{u}(t) =$ "$P \exp \int_0^t (-A(s)) ds$" $\underline{u}(0)$ which means that

$$u(t) = \left(\sum_{n \geq 0} \int_{\Delta^n := 0 \leq t_1 \leq t_2 \cdots \leq t_n} (-A(t_n))(-A(t_{n-1})) \dots (-A(t_1)) dt_1 \dots dt_n \right) \underline{u}(0)$$

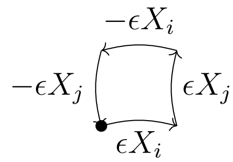
Here $A(t_n) = A(\dot\gamma(t))$.

In any case, the existence of a solution to (12.1) implies that we can define a linear map

$$E_p \to E_q$$
$$e_p \eqqcolon \underline{u}(0) \mapsto \underline{u}(1)$$

called parallel transport along a path γ from p to q by solving (12.1) for $\underline{u}(t)$ with $u(0) = e_p \in E_p$.

12.3. **Curvature.** Geometrically, curvature measures the deformation of infinitesimal parallel transport. In fact, the holonomy $E_p \to E_p$ around small "rectangle" defined by vectors X_i and X_j can be written as $\mathrm{id} + \epsilon^2 F(X_i, X_j)$ for $F \in \Omega^2(M, \operatorname{End} E)$. The F is known as the curvature 2-form.



In differentiail topological language, curvature is defined by extending covariant derivation to all of $\Omega^*(M, E)$ by making it a derivation with respect to $\Omega^*(M;\mathbb{C})$: In fact, define

$$\nabla\left(\sum \alpha_i \otimes s_i\right) := \sum_i d\alpha_i \otimes s_i + \sum \alpha_i \wedge \nabla s,$$

for any $\alpha_i \in \Omega^p(M)$ and $\sum \alpha_i \otimes s_i \in \Omega^p(M, E) = \Gamma(\Omega^p(M) \otimes E)$.

Theorem 12.7. *The map*

$$\nabla \circ \nabla : \Omega^p(M, E) \to \Omega^{p+2}(M, E)$$

is an algebraic operator given by $F \wedge (\bullet)$ for a two form $F \in \Omega^2(M, \operatorname{End} E)$. In local coordinates,

$$F = dA + A \wedge A,$$

where A is the connection one form in $\Omega^1(U, E)$. Furthermore, in local coordinates, under th egauge transformation

$$A \mapsto g^{-1}dg + g^{-1}Ag$$

the curvature form changes to

$$F \mapsto g^{-1}Fg.$$

12.4. **Chern–Weil Theorem.**

Theorem 12.8 (Chern–Weil)**.** *Fix a $\mathbb{C}^k$-bundle.*

$$\begin{tikzcd} \mathbb{C}^k \arrow[r] & E \arrow[d] \\ & M \end{tikzcd}$$

Let ∇ be a connection and $F_\nabla \in \Omega^2(\operatorname{End} E)$ be the associated curvature form. Then

$$\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j \in \Omega^{2j}(M;\mathbb{C})$$

is a closed $2j$-form on M whose cohomology class is independent of the choice of ∇ giving rise to a topological invariant of E lying in $H^{2j}(M;\mathbb{C})$.

Remark 12.9. We need to unpack what $\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j$ means. First,

$$(F_\nabla)^j \coloneqq \underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{j}.$$

To understand the wedge product, recall that given two bundles E, F over M, the wedge product is defined as

$$\begin{gathered}\wedge : \Omega^p(E) \otimes \Omega^q(F) \to \Omega^{p+q}(E \otimes F)\\ (\sum \alpha_i \otimes s_i, \sum \beta_j \otimes t_j) \mapsto \sum (\alpha_i \wedge \beta_j) \otimes (s_i \otimes t_j).\end{gathered}$$

Further, given vector bundles E, F, H over M, the bundle map $E \otimes F \to H$ induces a map

$$\Omega^*(M; E \otimes F) \to \Omega^*(M; H).$$

In our case, we get a wedge product

$$\wedge : \underbrace{\Omega^2(\operatorname{End} E) \otimes \cdots \otimes \Omega^2(\operatorname{End} E)}_{p} \to \Omega^{2p}(\operatorname{End} E \otimes \cdots \otimes \operatorname{End} E).$$

On the other hand, there is a bundle map

$$\operatorname{End} E \otimes \cdots \otimes \operatorname{End} E \xrightarrow{\times} \operatorname{End} E \xrightarrow{\operatorname{tr}} \mathbb{C},$$

where $\times$ is the multiplication map of matrices on the fiber of $\operatorname{End} E$ viewed as a principal $GL(k, \mathbb{C})$ bundle with the fundamental representation of $GL(k, \mathbb{C})$ on $\mathbb{C}^k$. Similarly, tr is the trace map on $\operatorname{End} E$ on the fiber of $\operatorname{End} E$ viewed as a principal $GL(k, \mathbb{C})$ bundle with the fundamental representation of $GL(k, \mathbb{C})$ on $\mathbb{C}^k$. Equivalently, we can think of $\operatorname{End} E \cong E^* \otimes E$ and tr is the natural pairing $E^* \otimes E \xrightarrow{\langle,\rangle} \mathbb{C}$ as a bundle map to the trivial bundle $\mathbb{C}$. More precisely, a vector bundle $E = P \times_G V$ where P is a principal G-bundle fibered with a representation V of G. Here, we can think of P as the frame bundle $\operatorname{Fr}(E)$ whose fiber consists of frames of fibers of E, $G = \operatorname{Gl}(k, \mathbb{C})$ and $V = \mathbb{C}^k$. where. Then $E^* \otimes E \xrightarrow{\langle,\rangle} \mathbb{C}$ is the map

$$(P \times_G V^*) \otimes (P \times_G V) \to P \times_G \mathbb{C}$$

induced by the natural pairing $V^* \otimes V \xrightarrow{\langle,\rangle} \mathbb{C}$.

In any case, the bundle map

$$\operatorname{End} E \otimes \cdots \otimes \operatorname{End} E \xrightarrow{\times} \operatorname{End} E \xrightarrow{\operatorname{tr}} \mathbb{C},$$

composed with $\wedge$ will induce a map

$$\underbrace{\Omega^2(\operatorname{End} E) \otimes \cdots \otimes \Omega^2(\operatorname{End} E)}_{p} \xrightarrow{\operatorname{tr}()^j} \Omega^{2j}(M; \mathbb{C}),$$

which we use to interpret $\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j$.

Remark 12.10. Choosing connections ∇ on G-bundles over M induces connections on all the associated bundles.

(1) Suppose ∇_E, ∇_F are connection on G bundles E and F respectively then
$$\nabla_{E\otimes F} = \nabla_E \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_F$$
is a connection on $E \otimes F$.

(2) Given ∇_E on E, we get a connection ∇_{E^*} on E^*:
$$\nabla_{E^*}(\phi)(s) \coloneqq (-1)^{\deg\phi}(-\phi \circ \nabla_E(s)),$$
where $\phi \in \Gamma(E^*)$ and $s \in \Gamma(E)$.

Given ∇_E on E, we get a connection $\nabla_{E^*\otimes E}$ on $E^* \otimes E$:
$$\nabla_{E^*\otimes E}(\psi) \coloneqq \nabla_E \otimes \psi - (-1)^{\deg\psi}\psi \circ \nabla_E,$$
where $\psi \in \Gamma(E^* \otimes E) \cong \Gamma(\operatorname{End} E)$.

Lemma 12.11 (Bianchi identity)**.** *Fix a connection ∇_E on a $\mathbb{C}^k$ vector bundle E over M and the associated curvature form $F_{\nabla_E} \in \Omega^2(M; \operatorname{End} E)$. Then*
$$\nabla_{\operatorname{End} E} F_E = 0. \tag{12.3}$$

Proof. Since $F = \nabla_E \circ \nabla_E$
$$\nabla_{\operatorname{End} E} F = \nabla_E \circ F - F \circ \nabla_E = 0.$$
□

Lemma 12.12 (Commutativity of trace map and derivative)**.** *If $\alpha \in \Omega^p(\operatorname{End} E)$ then*
$$d \operatorname{tr} \alpha = \operatorname{tr}(\nabla_{\operatorname{End} E}\alpha).$$

Proof of Theorem 12.8. To prove the closedness of the form, we can use Lemma 12.12, Leibniz rule and the Bianchi identity to get
$$d \operatorname{tr}(F^k) = \operatorname{tr}(\nabla_{\operatorname{End} E}(F^k)) = \operatorname{tr}(\nabla_{\operatorname{End} E} F \wedge F^{k-1} + F \wedge \nabla_{\operatorname{End} E} F \wedge F^{k-2} + \dots) = 0.$$

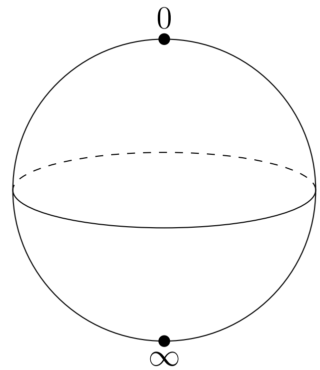
To prove that $\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j$ does not depend on ∇ consider the variation $\nabla_t \coloneqq \nabla + t\alpha$ where $\alpha \in \Omega^1(M; \operatorname{End} E)$. Let $F_t \coloneqq F_{\nabla_t}$. Then
$$\left.\frac{d}{dt}\right|_{t=0} F_t = \left.\frac{d}{dt}\right|_{t=0} (\nabla + t\alpha) \circ (\nabla + t\alpha) = \alpha \circ \nabla + \nabla \circ \alpha = \nabla_{\operatorname{End} E}\alpha.$$
Thus, using the fact that trace is graded cyclic, i.e., $\operatorname{tr}(\alpha \wedge \beta) = (-1)^{\deg\alpha \deg\beta} \operatorname{tr}(\beta \wedge \alpha)$, the Bianchi identity and Lemma 12.12, we get
$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (\operatorname{tr} F_t^k) &= \operatorname{tr}\left(\left.\frac{d}{dt}\right|_{t=0} F_t \wedge F^{k-1} + F \wedge \left.\frac{d}{dt}\right|_{t=0} F_t \wedge F^{k-2} + \dots\right) \\
&= \operatorname{tr}\left(\nabla_{\operatorname{End} E}\alpha \wedge F^{k-1} + F \wedge \nabla_{\operatorname{End} E}\alpha \wedge F^{k-2} + \dots\right) \\
&= k \operatorname{tr}(\nabla_{\operatorname{End}}\alpha \wedge F^{k-1}) \\
&= k \operatorname{tr} \nabla_{\operatorname{End}}(\alpha \wedge F^{k-1}) \\
&= d[k \operatorname{tr}(\alpha \wedge F^{k-1})].
\end{aligned}$$
Therefore, the cohomology class defined by $\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j$ remains constant. □

Example 12.13. We will show that $\mathrm{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)$ for the tautological line bundle L over $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is negative of the generator of $H^2(\mathbb{P}^1;\mathbb{C})$.

Explicitly, the fiber of L at $\zeta \in \mathbb{C} \cup \{\infty\}$ is

$$L_\zeta = \left\{(z,w) : \frac{z}{w} = \zeta\right\} \subset \mathbb{C}^2.$$

We view $\mathbb{P}^1$ as the Riemann sphere.



Locally,

$$U = \{\zeta \neq \infty\}, \text{ basis section } s_U(\zeta) = (\zeta, 1)$$
$$V = \{\zeta \neq 0\}, \text{ basis section } s_V(\zeta) = (1, \zeta^{-1}).$$

We define a connection locally. On the lower hemisphere (that is not all of V), pick a connection with connection form 0, i.e, $\nabla s_V = 0$. Then on $U \cap V$, we have

$$s_U(\zeta) = \zeta s_V(\zeta),$$

i.e., the transition function is $g_{UV}(\zeta) = \zeta$. Therefore,

$$\nabla s_U = \nabla(\zeta s_V) = (d\zeta)s_V + \zeta \nabla s_V = (d\zeta)\zeta^{-1}s_U.$$

In particular, the connection form on U is $A_\zeta = (d\zeta)\zeta^{-1}$. Thus, on the unit circle $\zeta = e^{i\theta}$

$$A_{e^{i\theta}} = ie^{i\theta}d\theta e^{-i\theta} = id\theta.$$

We can extend this connection form over the upper hemisphere (unit disc) by picking any smooth extension of the form $id\theta$ on the unit circle. For example, check that $\tilde{A} = ir^2 d\theta$ is smooth by noting that $d\theta = \frac{ydx - xdy}{x^2+y^2}$.

The curvature form on the unit disc is $d\tilde{A}$, so when we integrate we get

$$-\int_{\mathbb{P}^1} \frac{1}{2\pi i}F = -\int_{B^2} \frac{1}{2\pi i}d\tilde{A} = \int_{\partial B^2} -\frac{1}{2\pi i}d\theta = -1.$$

In particular,

$$\mathrm{tr}\left(-\frac{1}{2\pi i}F_\nabla\right) = -1 \in H^2(\mathbb{P}^1;\mathbb{Z}) \subset H^2(\mathbb{P}^1;\mathbb{C}).$$

12.5. **Chern class and Chern character.** Given a bundle with structure group $GL(k,\mathbb{C})$, any *invariant polynomial* on the Lie algebra $\mathfrak{g} = \mathfrak{gl}(k,\mathbb{C}) = k \times k$ matrices will give us a *characteristic class* just like how $\operatorname{tr}\left(-\frac{1}{2\pi i}F_\nabla\right)^j$ defines a cohomology class.

Definition 12.14. An invariant polynomial p is a map

$$p : \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \to \mathbb{C}$$

invariant under the action of the group $GL(k,\mathbb{C})$.

Example 12.15. (1) In Theorem 12.8, we used the invariant polynomial $s_j = \{X \mapsto \operatorname{tr}(X^j)\}$.
(2) The map $\sigma_j = \{X \mapsto \operatorname{tr}(\Lambda^j X)\}$ defines an invariant polynomial.

Theorem 12.16. *For $\mathfrak{g} = \mathfrak{gl}(k,\mathbb{C}) = k \times k$, the ring of invariant polynomials is generated (over $\mathbb{Q}$) by either $s_1, \dots, s_k$ or $\sigma_1 \dots \sigma_k$.*

Proof. To understand the basis of invariant polynomials, it suffices to consider open dense set of $\mathfrak{g}$. note that almost all matrices (i.e., open dense set) are diagonalizable via conjugation by $GL(k,\mathbb{C})$ to $diag(\lambda_1, \dots, \lambda_k)$. Since invariant polynomials are invariant under conjugation, for diagonalizable matrices, invariant polynomials are just symmetric polynomials in the λ_i's. On the other hand, it's well known that symmetric polynomials are generated by elementary symmetric functions:

$$\sigma_1 = \sum \lambda_i, \quad \sigma_2 = \sum_{i_1 < i_2} \lambda_{i_1}\lambda_{i_2}, \quad \dots, \quad \sigma_k = \lambda_1 \dots \lambda_k,$$

or by the power sums

$$s_1 = \sum \lambda_i, \quad s_2 = \sum \lambda_i^2, \quad \dots, \quad s_n = \sum \lambda_i^n.$$

□

Remark 12.17. Newton polynomials relate the two bases of invariant polynomials. For instance, $\sigma_2 = 1/2(s_1^2 - s_2)$ etc.

Definition 12.18. The *Chern character* $\operatorname{ch}(E)$ of a vector bundle E is the characteristic class associated to the power series of the invariant polynomial

$$\operatorname{ch}(E) \coloneqq \operatorname{tr}(e^{-\frac{1}{2\pi i}F}) \coloneqq \sum_k \frac{1}{k!}\left(-\frac{1}{2\pi i}\right)^k \operatorname{tr}(F^k) \in H^{even}(M;\mathbb{C}).$$

Definition 12.19. The *total Chern class* $c(E)$ is the characteristic class associated to the polynomial

$$c(E) = \det\left(1 - \frac{1}{2\pi i}F\right).$$

Here, $\det\left(1 - \frac{1}{2\pi i}F\right)$ means that for diagonal matrix

$$\det(1 + \operatorname{Diag}(\lambda_1, \dots, \lambda_k)) = \sigma_1 + \cdots + \sigma_k$$

corresponding to $\sum_j \operatorname{tr}(\Lambda^j \operatorname{Diag}(\lambda_1, \dots, \lambda_k))$. We can also write

$$\det\left(1 - \frac{1}{2\pi i}F\right) = \sum \left(\frac{-1}{2\pi i}\right)^j \operatorname{tr}(\Lambda^j F) \in H^{even}(M; \mathbb{C}).$$

Remark 12.20.
(1) It is not hard to see that for two vector bundles E and F over M,

$$\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F)$$
$$\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \cup \operatorname{ch}(F).$$

Therefore, ch is a ring homomorphism from the ring of vector bundles to the ring of cohomology ring of M.

(2) Whereas the total Chern class is multiplicative for sum of bundles.

$$c(E \oplus F) = c(E) \cup c(F).$$

It is much harder to see that the total Chern class is an integral class i.e., $c(E) \in H^{even}(M; \mathbb{Z})$ and $\operatorname{ch}(E) \in H^{even}(M; \mathbb{Q})$.

(3) We can treat characteristic classes entirely topologically and not differential topologically. See [MS74]. Closer to our approach is the book of Taubes [Tau11].

12.6. **Chern–Simons form.** Consider a rank 2 $(SU(2))$[8] bundle with connection ∇

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & E \\ & & \downarrow \\ & & M \end{array}$$

on an arbitrary manifold M. We saw that $c_2(E) = [1/8\pi^2 \operatorname{tr}(F_\nabla \wedge F_\nabla)] \in H^4(M; \mathbb{Z})$ is a topological invariant looking at the variation $\nabla_t = \nabla + t\alpha$ where $\alpha \in \Omega^1(M; \operatorname{End} E)$ so that

$$\frac{d}{dt} \operatorname{tr}(F_t^2) = 2d \operatorname{tr}(\alpha \wedge F_t).$$

By integrating the above expression and noting that $F_t = F_\nabla + td\alpha + t^2\alpha \wedge \alpha$, we can write the first variation of $\operatorname{tr}(F_t \wedge F_t)$ as follows

$$\begin{aligned} \operatorname{tr}(F_1 \wedge F_1) - \operatorname{tr}(F_0 \wedge F_0) &= \int_0^1 2d \operatorname{tr}(\alpha \wedge (td\alpha + t^2\alpha \wedge \alpha))dt \\ &= d \operatorname{tr}(\alpha \wedge d\alpha + 2/3\alpha \wedge \alpha \wedge \alpha), \end{aligned}$$

which is equal to an explicitly co-boundary formula for the change in characteristic form. The expression on the right hand side bears a name.

Definition 12.21. The *Chern–Simons form* is a 3-form

$$\Omega^1(M; \operatorname{End} E) \ni \alpha \mapsto \operatorname{tr}(\alpha \wedge d\alpha + 2/3\alpha \wedge \alpha \wedge \alpha).$$

[8] In general we can take simply connected compact simple Lie group

12.7. **Chern–Simons functional.** Now suppose M is a closed oriented 3-manifold. On such manifold a rank 2 bundle E is trivialisable. In fact, such bundles are classified by $[M, BSU(2)]$, the homotopy classes of maps from M to the *classifying space* $BSU(2)$. Here, $BSU(2)$ is the base space of the fibration

$$\begin{tikzcd} SU(2) \arrow[r] & ESU(2) \arrow[d] \\ & BSU(2) \end{tikzcd}$$

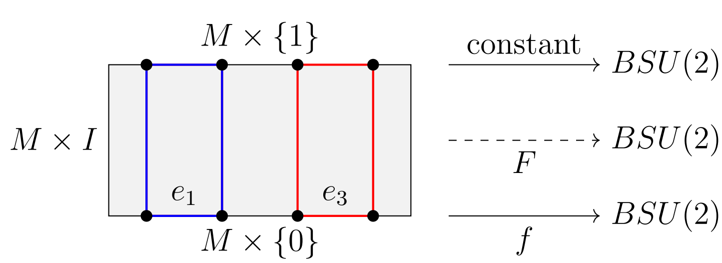
with contractible total space $ESU(2)$ and a free and transitive smooth action of $SU(2)$. The classifying space is the quotient of the action. More concretely, the above bundle can be thought of as

$$\begin{tikzcd} SU(2) \arrow[r] & \mathbb{H}^\infty - \{0\} \arrow[d] \\ & \mathbb{H}P^\infty, \end{tikzcd}$$

where $\mathbb{H}$ is the space of quaternions. It implies that

$$\pi_{i+1}(BSU(2)) = \pi_i(SU(2)) = 0$$

for all $i \leq 2$ since $SU(2)$ is just the three sphere.[9] In any case, by obstruction theory, there does not exist any obstruction to null-homotoping an arbitrary map $M \to BSU(2)$. The point is that given a map $f : M \times \{0\} \to BSU(2)$ and a constant map $M \times \{1\} \to BSU(2)$, we can find a map $F : M \times I \to BSU(2)$ by viewing $M \times I$ as a CW complex and doing the following procedure. For each vertex in M, there is a line in $M \times I$. Since $BSU(2)$ is path connected implies that we can extend the constant map and f on that line. Now given an edge e_1 in the CW decomposition of M and the lines which form an S^1 (blue line in the picture below) we get a map from S^1 to $BSU(2)$. Since $BSU(2)$ is simply connected it means that we can extend the map to the 2-cell that the S^1 bounds. Similarly, since $\pi_3(BSU(2)) = 0$ we can extend the map on a 3-cell and the its boundary times I (which gives rise to red S^3 in the picture below) to the 4-cell in $M \times I$. Since $M \times I$ is 4-dimensional, it means that we actually go an extension $F : M \times I \to BSU(2)$.



Therefore, the $SU(2)$-bundle E over M is trivializable.

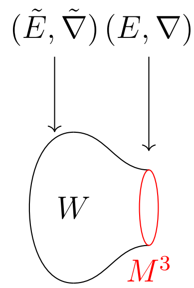
[9]The above computation works also for simply connected Lie group.

Further, every such M^3 bounds a compact oriented 4-manifold, say W^4. This follows because the *cobordism group* is trivial, i.e.,

$$\Omega_3^{SO} = 0.$$

Remark 12.22. The triviality of cobordism group in dimension three is a theorem due to Rokhlin and later due to Lickorish–Wallace. In dimension four, the signature of a manifold is obstruction for it to be boundary of a five dimensional manifold.

Since the (E, ∇) is trivializable, it can be extended over to W to get a bundle $(\overline{E}, \overline{\nabla})$. Namely, if we choose a trivialization of E then $\tilde{E}$ can be taken to be a trivial bundle over W.

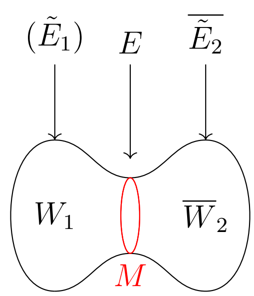


Define

$$CS(\nabla) := \frac{1}{8\pi^2} \int_W \mathrm{tr}(F_{\tilde{\nabla}} \wedge F_{\tilde{\nabla}}).$$

Lemma 12.23. *$CS(\nabla)$ is a number well defined in $\mathbb{R}/\mathbb{Z}$ independent of choice of W and extensions of E and of ∇.*

Proof. Take two such extension W_1 and W_2 and glue the reverse of one to other.



Then if we denote F to be the curvature of the connection associated to the bundle over $W_1 \cup \overline{W}_2$ then

$$CS(\nabla_1) - CS(\nabla_2) = \frac{1}{8\pi^2} \int_{W_1 \cup \overline{W}_2} \mathrm{tr}(F \wedge F) = c_2(\text{bundle over } W_1 \cup \overline{W_2}) \in \mathbb{Z}.$$

$\square$

Remark 12.24. (1) In the proof, there might be some non-trivial twisting where things are glued along M. Namely, bundle over $W_1 \cup \overline{W_2}$ might be non-trivial.
(2) The Lemma implies that $CS(\nabla)$ is independent under gauge transforms.

(3) However, if we pick a trivialization of $E \to M$ and of $\tilde{E} \to W$, we induce a trivial connection $\tilde{\nabla}_0$ on W. Hence we could write any connection $\tilde{\nabla} = \tilde{\nabla}_0 + \alpha$ for some one form α. Then

$$\begin{aligned} CS(\nabla) &= \frac{1}{8\pi^2}\int_W \operatorname{tr}(F_{\tilde{\nabla}} \wedge F_{\tilde{\nabla}}) \\ &= \frac{1}{8\pi^2}\int_W \operatorname{tr}(F_{\tilde{\nabla}} \wedge F_{\tilde{\nabla}}) - \operatorname{tr}(F_{\tilde{\nabla}_0} \wedge F_{\tilde{\nabla}_0}) \\ &= \frac{1}{8\pi^2}\int_M \operatorname{tr}(\alpha \wedge d\alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha), \end{aligned}$$

where the second line follows because $\tilde{\nabla}_0$ is a flat connection and the third line follows because of Stoke's theorem and the fact that $\operatorname{tr}(F_1 \wedge F_1) - \operatorname{tr}(F_0 \wedge F_0) = d\operatorname{tr}(\alpha \wedge d\alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha)$.

Note that α is not a canonical thing associated to the connection ∇. It depends on a reference trivialization $\tilde{\nabla}_0$. Changing trivialization (which amounts to gauge transformation) changes α but we have seen that $CS(\nabla)$ does not change under gauge transformation. We can also prove this fact by explicit computation after changing $\alpha \mapsto g^{-1}\alpha g + g^{-1}dg$.

Definition 12.25. The *Chern–Simons action* is $CS(\nabla) \in \mathbb{R}/\mathbb{Z}$.

From now onwards, we write α as A.

Lemma 12.26 (First variation formula)**.** *Take $a \in \Omega^1(M; \operatorname{End} E)$. Then*

$$CS(A + ta) - CS(A) = 2t\int_M \operatorname{tr}(a \wedge (dA + A \wedge A)) + \mathcal{O}(t^2).$$

In particular, if $CS(A+ta) - CS(A)$ vanishes for all a if and only if

$$dA + A \wedge A = 0,$$

which is true if and only if A is flat.

Proof. Note that the Leibniz rule on the definition of the Chern–Simons form implies that

$$CS(A + ta) - CS(A) = t\int_M \operatorname{tr}(a \wedge dA + A \wedge da + 2A \wedge A \wedge a) + \mathcal{O}(t^2).$$

Further, using the signed Leibniz rule we have,

$$\int_M d(\operatorname{tr}(A \wedge a)) = \int_M \operatorname{tr}(dA \wedge a) - \operatorname{tr}(A \wedge da).$$

But the left hand side vanishes because M is closed. Therefore,

$$CS(A + ta) - CS(A) = 2t\int_M \operatorname{tr}(a \wedge (dA + A \wedge A)) + \mathcal{O}(t^2).$$

□

The above Lemma implies that the critical points of the Chern–Simon functional CS on

$$\mathcal{A}/G = \text{space of all connections on } M$$

are the *flat* connections.

13. Combinatorial recipe for getting three manifold invariants

After Witten's paper, mathematicians were able to completely describe the TQFT by alternative combinatorial method. Reshetikin and Turaev used quantum groups representations. Lickorish did it usign Kauffmann bracket for $SU(2)$.

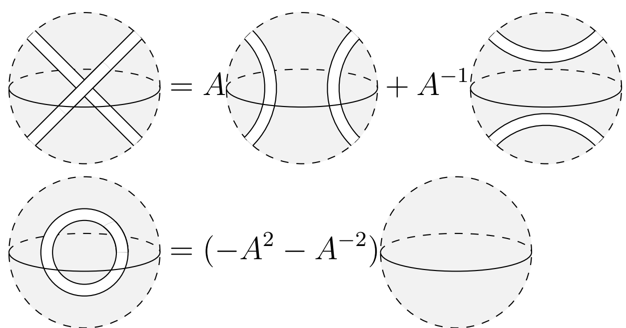
The goal for the rest of the notes is to construct 3-manifold invariants and a TQFT from combinatorial methods, starting from Kauffman bracket (which is just a framed version of Jones polynomial).

13.1. **Skein space.**

Definition 13.1. For a compact three manifold with boundary $(M^3, \partial M = \Sigma)$, if we fix some framed points "c" in Σ, we define

$$\mathcal{S}(M,c) = \frac{\mathbb{Z}[A^{\pm}] - \text{linear combinations of framed links in } M \text{ with } \partial = C \text{ in } \Sigma}{\text{framed isotopy (relative to } \partial) \text{ Kauffman relations}}.$$

Here, the Kauffman relations are



Going forward, we won't draw the framing. Also, the shading emphasizes that the snapshot is of B^3. We will also denote the empty link as $[\emptyset]$.

Remark 13.2. In the above Kauffman relations, we don't have a normalization such that the Kauffman bracket evaluation of an unknot 1.

Example 13.3. $\mathcal{S}(S^3) \cong \mathbb{Z}[A^{\pm 1}][\emptyset]$ with basis element the empty link. Here, the isomorphism is given by

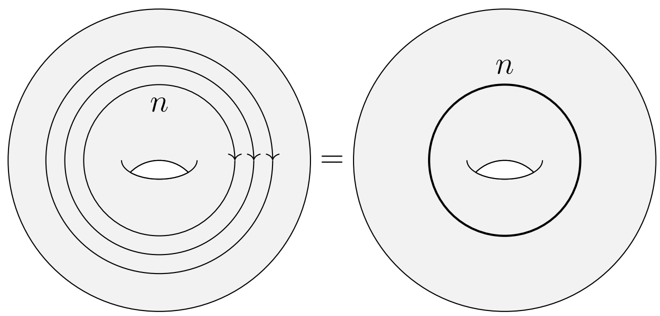
$$\mathcal{S}(S^3) \ni [L] \mapsto \langle L \rangle [\emptyset]$$

where $\langle L \rangle$ is renormalized so that the unknot $= (-A^2 - A^{-2})$. The map sends every framed link $[L]$ to its Kauffmann bracket evaluation in the normalization equals unknot $= (-A^2 - A^{-2})$. The fact that invariance of links under Reidemeister moves type isotopies makes the isomorphism of $\mathcal{S}(S^3) \cong \mathbb{Z}[A^{\pm 1}]$ a non-trivial and is essentially the existence result of the Kauffman bracket.

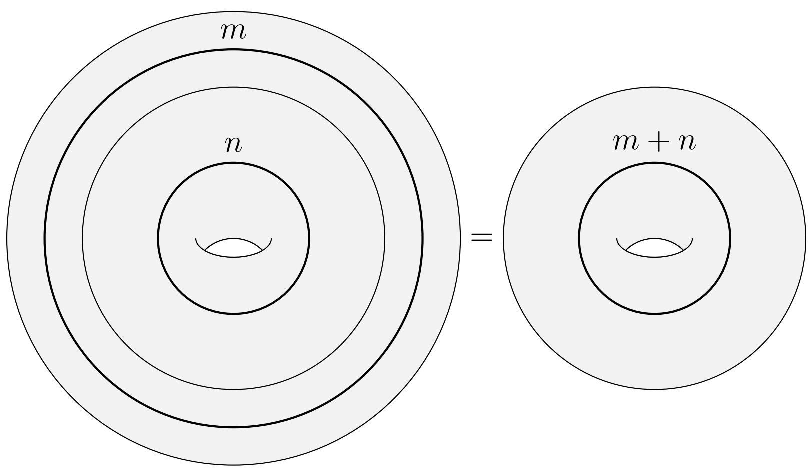
Remark 13.4. More complicated manifold might not have very good algebraic structures because we might not be able to compose them in natural ways.

Example 13.5. $\mathcal{S}(S^1 \times B^2) \cong \mathbb{Z}[A^{\pm 1}][\alpha]$.

After using the Kauffmann relations repeatedly, we can get rid of all the crossings on a knot inside $S^1 \times B^2$. Similar, we can also remove any unknot that can contract. The result will be to get n unknots that represent non-trivial homology of $S^1 \times B^2$. We will represent them as follows:



$\mathcal{S}(S^1 \times B^2)$ has a natural (bilinear) algebra structure $\mathcal{S}(S^1 \times B^2) \times \mathcal{S}(S^1 \times B^2) \to \mathcal{S}(S^1 \times B^2)$ got by concentric juxtaposition. Namely, we can glue a torus with m strings and n strings to get a torus with $m + n$ strings. The planar view is as follows:

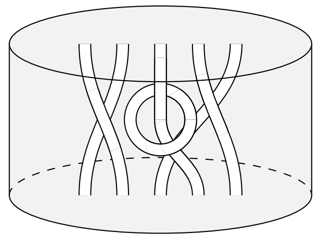


Using this algebra structure, we can see that the element α with $n = 1$ is the generator of $\mathcal{S}(S^1 \times B^2)$ over the ground ring $\mathbb{Z}[A^{\pm 1}]$.

Remark 13.6. In general, if we have N^3 with framed boundary M^3 with framed points in boundary then gluing them along the boundary along some framed points to get (W^3, c) then we will get a bilinear map $\mathcal{S}(M^3, c) \times \mathcal{S}(N^3, c') \to \mathcal{S}(W, c')$.

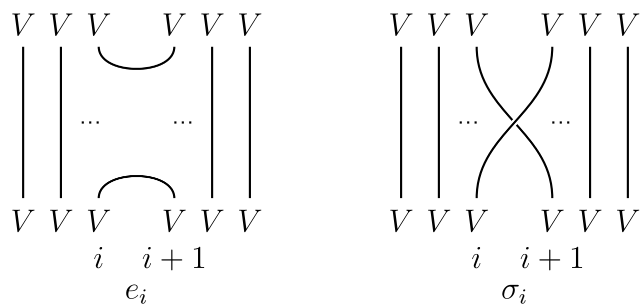
Example 13.7. $\mathcal{S}(B^2 \times I, n + n \text{ points in } \partial) = TL_n$, the Temperly–Lieb algebra.

For instance, an element of $\mathcal{S}(B^2 \times I, 5 + 5)$ would look like the following.



Note that modulo the Kauffman relations, we can view the above object in planar diagrams on a rectangle modulo the same Kauffman relations which is just the TL_n. Recall that TL_n is generated by $\langle e_1, \ldots, e_{n-1}\rangle$ with relations like $e_i^2 = (-A^2 - A^{-2})e_i$. The algebra structure is obtained by vertical Juxtaposition. Further, $\dim TL_n = \frac{1}{n+1}\binom{2n}{n}$.

Secretly (from quantum group point of view) we are looking at the invariant (under $U_{q\mathfrak{sl}(2)}$) homomorphisms $\mathbf{Hom}_{U_{q\mathfrak{sl}(2)}}(V^{\otimes n}, V^{\otimes n})$ from the fundamental representation V of $U_{q\mathfrak{sl}(2)}$. The tensor category of representation is the category of equivariant maps from $V^{\otimes n}$ to $V^{\otimes n}$. The e_i are generators of these equivariant maps in $\mathbf{Hom}_{U_{q\mathfrak{sl}(2)}}(V^{\otimes n}, V^{\otimes n})$. See the picture below.

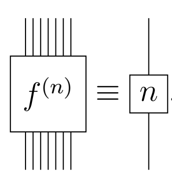


We are doing a quantum version of Schur–Weyl duality which says that the equivariant maps $\mathbf{Hom}_{\mathfrak{gl}(k)}(V^{\otimes n}, V^{\otimes n})$ are generated by the group algebra of the symmetric groups consisting of permutations σ_i. We should think of e_i as a quantum (degenerated) version of σ_i.

Inside this algebra is an idempotent $f^{(n)}$ which corresponds to the projection to S^nV inside $V^{\otimes n}$. In the classical representation theorem, the

$$f^{(n)} = [V^{\otimes n} \xrightarrow{\pi} S^nV \xhookrightarrow{\iota} V^{\otimes n}].$$

We can build this idempotent explicitly in TL_n These are called *Jones–Wenzl* idempotents. The notations for f are the following:



Theorem 13.8. *Fix $A \in \mathbb{C}$ such that $A^{4(r-1)} \neq 1$. Then we can construct elements*

$$f^{(n)} \in TL_n \textit{ for } n = 0, 1, \ldots, r-1$$

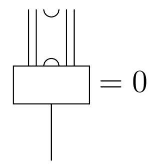
having properties

(1)

$$f^{(n)} = 1_n + (\textit{something in the ideal generated by } e_i).$$

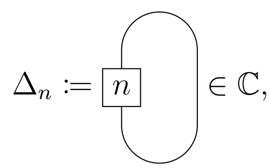
Ideal of stuff with “backtracks.” Here, 1_n consists of n parallel strings

(2) $f^{(n)}$ *annihilates the* e_i*'s for all* $i = 1, \ldots, n-1$, *i.e.,* $f^{(n)} e_i = 0 = e_i f^{(n)}$.



(3) $f^{(n)} f^{(n)} = f^{(n)}$.

(4) *If we define*[10]



Then

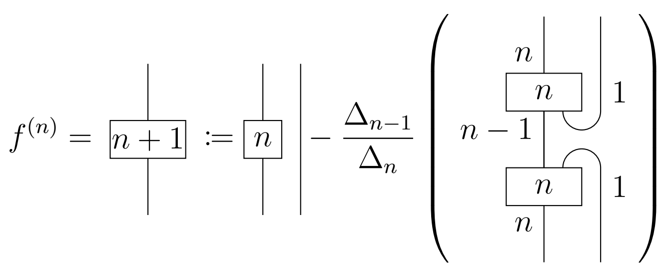
$$\Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}.$$

Proof. The proof is by induction on n. Set

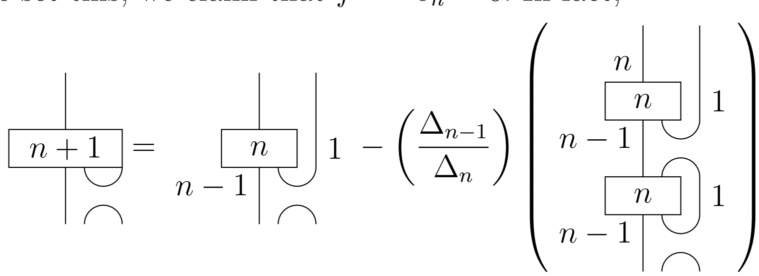
$$f^{(0)} = \text{empty picture}, \quad \Delta_0 = 1,$$

$$f^{(1)} = \Big|, \quad \Delta_1 = -A^2 - A^{-2}.$$

Inductively, we set

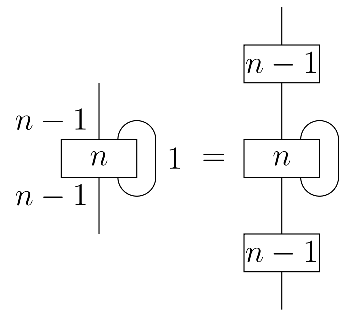


Once we set this, we claim that $f^{(n+1)} e_n = 0$. In fact,

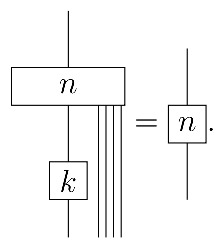


[10]Note that $\Delta_n \in \mathcal{S}(S^3) = \mathbb{Z}[A^{\pm 1}]$ but $A \in \mathbb{C}$, so $\Delta_n \in \mathbb{C}$.

Note that



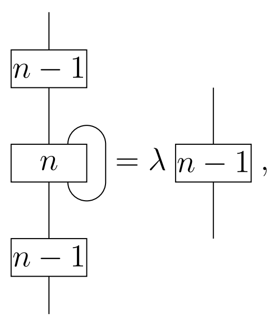
because $f^{(n-1)}$ annihilates the e_i's which implies



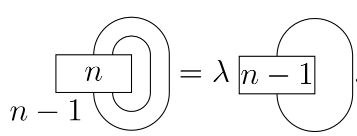
The last line follows because $f^{(k)}$ consists of 1_k and backtracks. The identity pass through but the backtracks get killed by $f^{(n)}$.

Using this computation again, we can see that

(13.1)



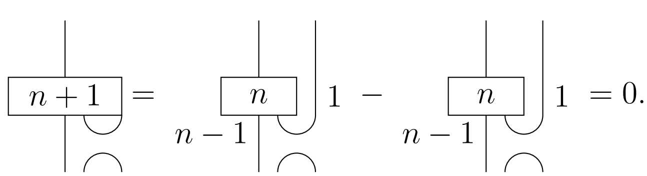
for some λ. We can calculate λ by closing all the diagrams up to get,



Now using the definition of Δ_n we know that

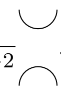
$$\lambda = \frac{\Delta_n}{\Delta_{n-1}}.$$

Therefore,



□

Example 13.9. For instance,

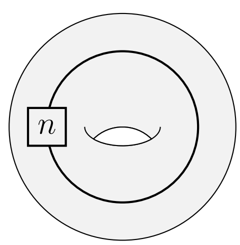
$$f^{(2)} = \Big|\,\Big| - \frac{1}{-A^2 - A^{-2}}$$

.

Remark 13.10. Since $\Delta_n \coloneqq (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$, we require A not to be a root of unity. Or A is a primitive $4r$-th root of unity then $f^{(0)}, \ldots, f^{(r-1)}$ exist and $\Delta_0, \ldots, \Delta_{r-2} \neq 0$ but $\Delta_{r-1} = 0$. In particular, $f^{(r-1)}$ annihilates everything when evaluated in $\mathcal{S}(S^3)$. The interpretation is that any closed diagram containing $f^{(r-1)}$ vanishes. In fact, after writing out things outside of $f^{(r-1)}$ in terms of basis of TL_n, we can write the closed diagram as Δ_{r-1} which vanishes by our construction. In other words, quantum dimension Δ_{r-1} of $f^{(r-1)}$ is 0.

In particular, when we do graphical computations and get diagrams containing $f^{(r-1)}$ then after closing them all up, the pieces with $f^{(r-1)}$ vanish.

On a parallel side, quantum groups also have similar property. The representation theory of quantum groups at root of unity is quite different from the generic case. This is essential in creating 3-manifold invariant because this is where the finiteness (of TQFT) comes from.

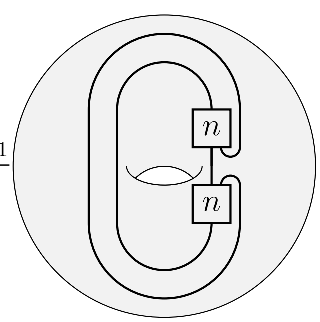
Example 13.11 (Solid torus revisited)**.** We will study the corresponding idempotents $f^{(r-1)}$ in $\mathcal{S}(S^1 \times B^2)$. Let

$\phi_n \coloneqq$


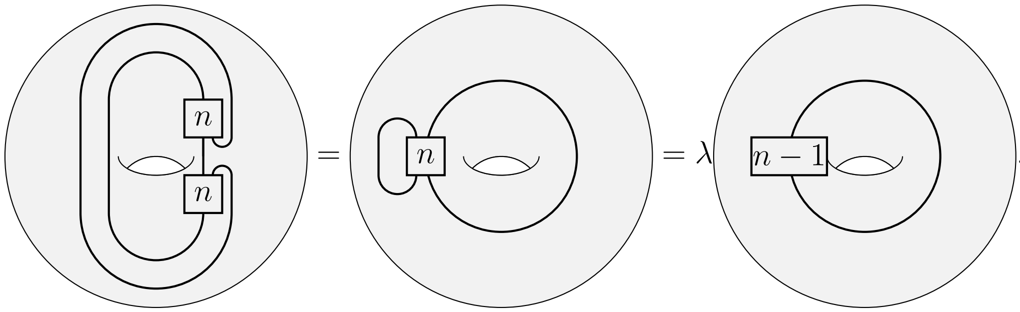
$\in \mathcal{S}(S^1 \times B^2) \cong \mathbb{C}[\alpha]$

where α is the generator with one string going around.

Using the inductive formula, we can see that

$$\phi_{n+1} = \alpha\phi_n - \frac{\Delta_{n-1}}{\Delta_n}$$


On the other hand, we can isotope the boxes towards the left and use the computation (13.1) to get



Using $\lambda = \frac{\Delta_n}{\Delta_{n-1}}$, we can conclude that

$$\phi_{n+1} = \alpha\phi_n - \phi_{n-1}.$$

We can use $\phi_0 = 1$ and $\phi_1 = \alpha$ to know ϕ_n inductively which will give us a polynomial on α.

In fact, the recursive implies that $\phi_n = S_n(\alpha)$, where S_n is a Chebyshev polynomial $\in \mathbb{Z}[\alpha]$ given by

$$S_{n+1}(\alpha) = \alpha S_n(\alpha) - S_{n-1}(\alpha)$$

with $S_0 = 1$ and $S_1 = \alpha$. For instance, $S_2 = \alpha^2 - 1$, $S_3 = \alpha^3 - 2\alpha$ and so on. But these are in fact be checked to satisfy

$$S_n(t + t^{-1}) = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}},$$

which after setting $t = e^{i\theta}$ gives

$$S_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

These really are giving us the irreducible representation of $SU(2)$ in its representation ring. Recall that the character of the fundamental representation V can be written as $\chi_V = e^{i\theta} + e^{-i\theta}$. Then the character of S^nV would be $\chi_{S^nV} = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta}$. The identity $\phi_{n+1} = \alpha\phi_n - \phi_{n-1}$ is really $V \otimes S^nV \cong S^{n+1}V \oplus S^{n-1}V$. Here $\alpha\phi_n$ corresponds to $V \otimes S^nV$. In particular, the idempotent are really the projectors inside the Temperately–Lieb algebra, which is a space of invariants for the quantum groups $U_{q\mathfrak{sl}(2)}$.

Remark 13.12. The skein of solid torus is the representation ring of the qunatum group $U_{q\mathfrak{sl}(2)}$. ϕ_n are the elements corresponding to the irreducible representation of $SU(2)$ as opposed to α that correspond to tensor powers of the fundamental representation. The Chebyshev polynomial gives us a change of basis with respect to ϕ_n and α.

Remark 13.13. The idempotent $f^{(n)}$ corresponds to projector to irreducible representation S^nV. Putting it inside the torus to get ϕ_n corresponds to the character of the representation.

Fix a primitive $4r$-th root of unity $A \in \mathbb{C}$. Let

$$\omega := \sum_{n=0}^{r-2} \Delta_n\phi_n \in \mathcal{S}^1(S^1 \times B^2).$$

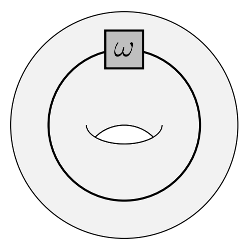
The element ω corresponds in the representation ring to what we might call (character of) "the regular representation" of the quantum group akin to

$$\sum_V V^{\oplus \dim V},$$

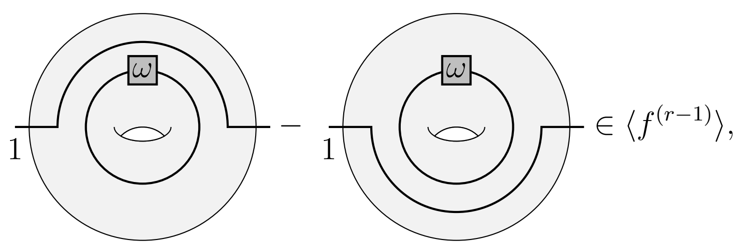
where V runs over all irreducible representations of $U_{q\mathfrak{sl}(2)}$. In the formula of ω, Δ_n is the "quantum dimension" of $S^n V$ of $U_{q\mathbf{sl}(2)}$ for $q = A^4$ and ϕ_n correspond to an irreducible representation $S^n V$. Classically, if we have a finite group, the regular representation is given by summing over irreducible representation, each one times its dimension.

Remark 13.14. Closing up an element X in tensor category (with cups and caps) gives us a scalar $\mathbf{Hom}_{\mathcal{C}}(1,1)$" $=\mathbb{C}$". This scalar is called the quantum dimension. Therefore, Δ_n is called the quantum dimension. Similarly, if we take any morphism from $X \to X$ then closing up the diagram gives us quantum trace. So the quantum trace gives us the quantum dimension.

Denote ω by

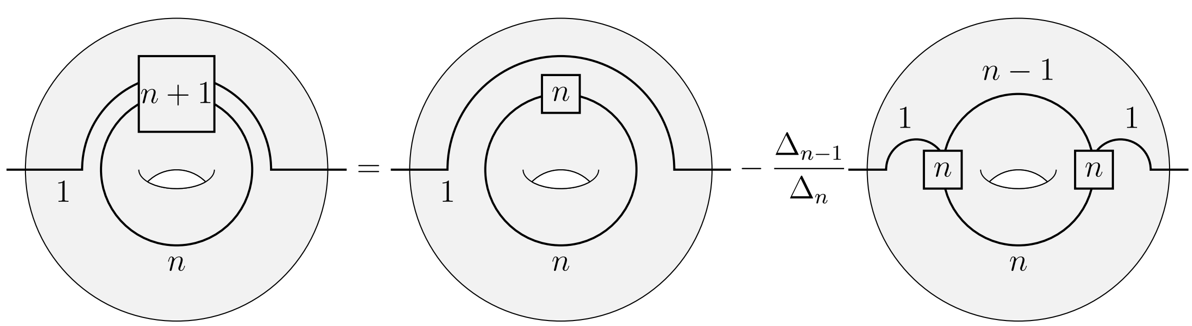


Lemma 13.15 (Fundamental lemma)**.** *In $\mathcal{S}(S^1 \times B^2, 2pts \in \partial)$, we have*

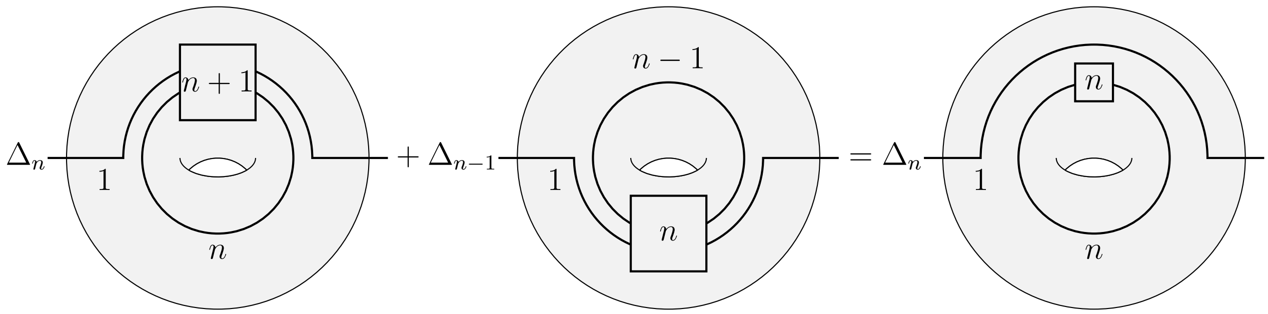


where $\langle f^{(r-1)}\rangle$ is the space generated by $f^{(r-1)}$. In particular, the element vanishes whenever evaluated by closing up in $\mathcal{S}(S^3)$.

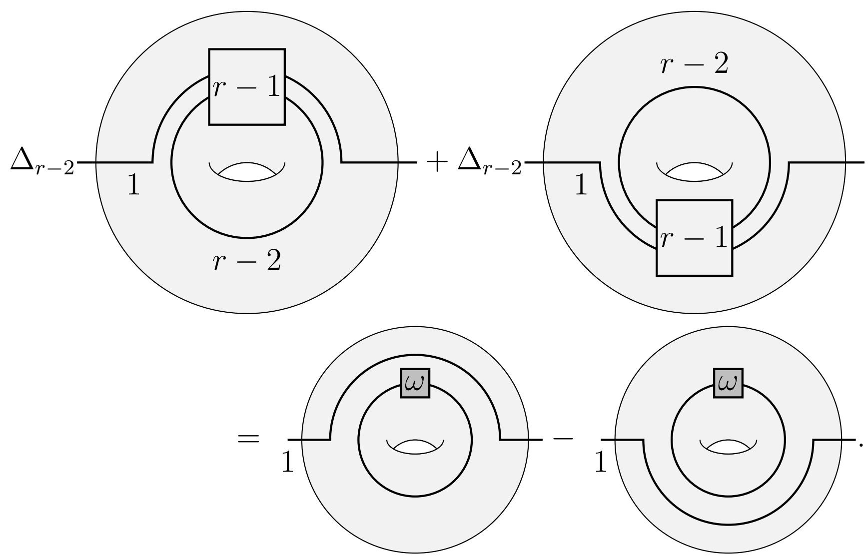
Proof. By inductive definition of idempotents we have,



Multiplying by Δ_n on both side and isotopying two n for the second term on the right and moving the term to the left, we get



Summing the above identity for $n = 0$ to $n = r - 2$ and take the difference between this and the same sum “all rotated by 180 degree, we get the following identity.

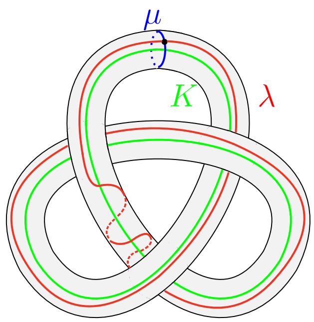


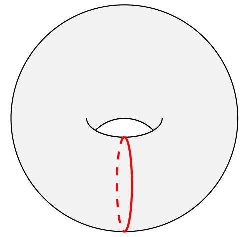
However, both graphics on the left hand sides are $f^{(r-1)}$. Therefore, the claim follows. $\square$

13.2. **Obtaining 3-manifolds by surgery on S^3.** We want to head towards obtaining three manifold invariants.

Theorem 13.16. *Every closed oriented 3-manifold can be obtained from S^3 by doing surgery on some framed link in S^3.*

Meaning, we remove a solid torus neighborhood of each component K of a framed link and glue back a torus differently so that ∂disc $\subset S^1 \times B^2$ goes to longitude λ, which is a curve deformed by framing. Recall that choosing a framing is equivalent to choosing a longitude up to isotopy.



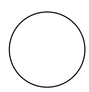


Pictorially, we draw framed links either with “blackboard framings” or by writing an integer next to each component, which denotes the linking number between the longitude and the core. In S^3 this makes sense. In general, if there is non-trivial first homology then the linking number is not necessarily defined.

The above is a bit like Reidemister theorem that prescribes how to get any three manifolds by pictures.

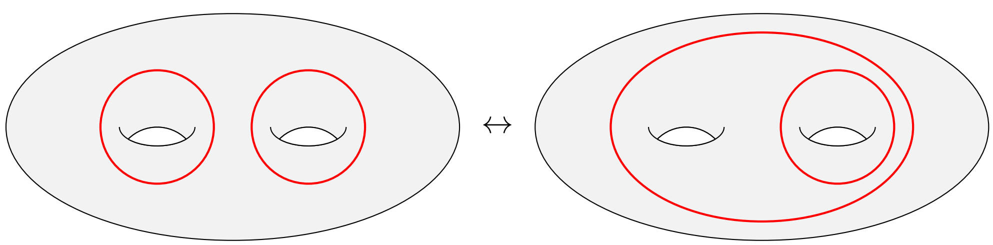
Theorem 13.17 (Kirby's theorem)**.** *Any two surgery description of the same three manifold differ by a finite sequence of Kirby moves of two types:*

(1) Type I: Introduce/delete a ± 1 framed unknot.

 $\pm 1 \leftrightarrow \emptyset$

Here, the number denotes the framing. It would have a ± 1 kink.

(2) Handleslide move: Consider a genus 2 handlebody embedded arbitrarily in S^3 containing two curves of framed links. Then we are allowed to change:

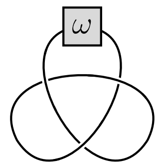


Remark 13.18. (1) Type I is an analog of Reidemeister I.
(2) Everything boils down to Morse theory.
(3) Handle slide in general is done by connect-summing the framing of two links.

The upshot is that if we “assign ω” to every component of a framed link L in S^3, we almost get an invariant of three manifolds built from surgery on the neighborhood of L. We denote such quantity by

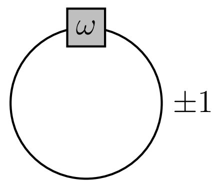
$$\langle \omega L \rangle$$

which denotes for example



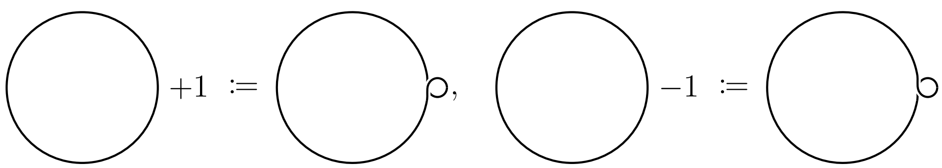
On the other hand, the exchange of two pictures in the Fundamental Lemma 13.15 is basically handle slide. Then Lemma 13.15 gives that after evaluation, the difference between handslide move between ω and ω' is 0 because $f^{(r-1)}$ evaluates to $\Delta_{r-1} = 0$.

Therefore, $\langle \omega L \rangle$ is invariant under the handle-slide move on L. However, it is not necessarily invariant under the Type I move. To understand the failure of invariance, we have to understand the value of the following diagram.



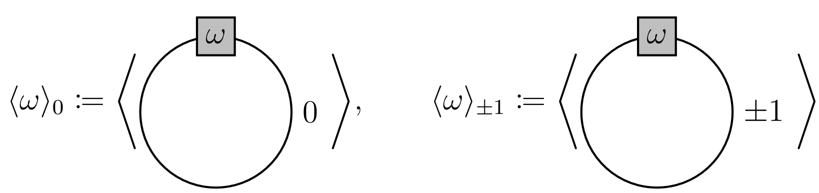
In fact, when we blow down an unknot or blow up to get an unknot, $\langle \omega L \rangle$ gets multiplied by the evaluation of the above diagram.

In our convention



and

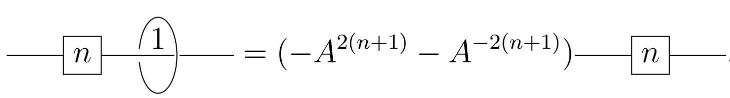
13.3. **Normalizations.** Let



By the definition of ω and Δ_n, we can see that

$$\langle \omega \rangle_0 = \sum_{n=0}^{r-2} \Delta_n^2 = \frac{-2r}{(A^2 - A^{-2})^2} =: \eta^{-2}$$

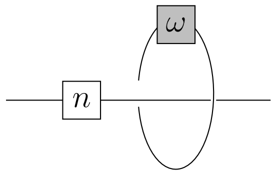
Then we can also use the following identities in TL_n after evaluation by skein relation:



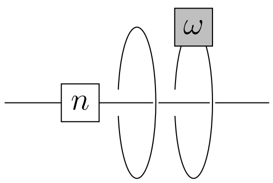
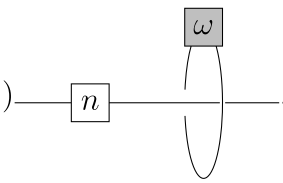
We used the fact that there are $2n$ crossings between f^n and the circle.

This gives us the following lemma.

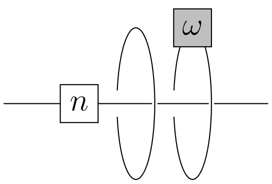
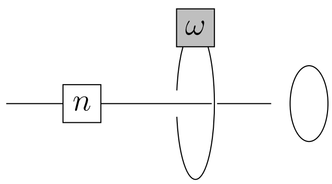
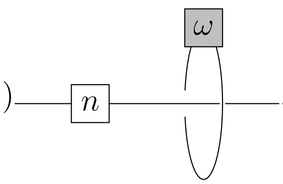
Lemma 13.19.


$$= \begin{cases} 0 & n \neq 0, \\ \eta^{-2} & n = 0. \end{cases}$$

Proof. First, we can use the above idenity to compute


$$= (-A^{2(n+1)} - A^{-2(n+1)})$$


On the other hand, the handle slide and the value of the unknot implies that


$$=$$

$$= (-A^2 - A^{-2})$$


□

Lemma 13.20.

$$\langle \omega \rangle_+ \cdot \langle \omega \rangle_- = \langle \omega \rangle_0 = \eta^{-2}.$$

Proof. Use handle slide and use the previous lemma. □

Define the renomalized version of ω

$$\Omega = \eta \cdot \omega$$

so that $\langle \Omega \rangle_+ \langle \Omega \rangle_- = 1$.

Remark 13.21. Observe that $\langle \Omega \rangle_+$ and $\langle \Omega \rangle_-$ are complex conjugates of one another. ω is a mirror image of the another. When we mirror image we inverse A. However, since A is a root of unity it its inverse is its conjugate. Therefore, $\langle \Omega \rangle_\pm$ are phases. Explicitly,

$$\langle \Omega \rangle_+ = \kappa = A^{-3-r^2} e^{i\frac{\pi}{4}}.$$

This uses Gauss sum (in number theory). We get sum of squares of roots of unity.

Define

$$I(M) := \kappa^{-\sigma(L)} \eta \langle \Omega L \rangle$$

η is normalization where $\sigma(L)$ is the signature of the linking matrix of L (with respect to any orientation of components and with framing numbers on diagonal).

$I(M)$ is like the Kauffmann bracket corrected with writhe account for its non-invariance under RI to get a Jones polynomial.

Theorem 13.22. *If M is a closed oriented* 3 *manifold obtained by surgery on a framed link $L \subset S^3$ then $I(M)$ is invariant of Kirby moves and hence an invariant of M.*

Proof. Observe that $\sigma(L)$ changes by ± 1 when we do the first Kirby move. In fact, if M_L is the linking matrix corresponding of blow down and $M_{L'}$ of the blow up them

$$M_{L'} = \begin{pmatrix} 1 & 0 \\ 0 & M_L \end{pmatrix}.$$

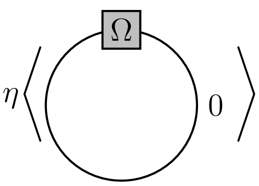
This change cancels out the value

$$\langle \Omega \rangle_{\pm} = \kappa^{\pm 1}$$

coming form the skein evaluation. □

Example 13.23. (1) $I(S^3) = \eta$ because S^3 can be obtained from S^3 by setting $L = \emptyset$.

(2) Note that we can get $S^1 \times S^2$ by gluing two solid tori by identifying the boundary. This is actually a 0-surgery. Therefore,

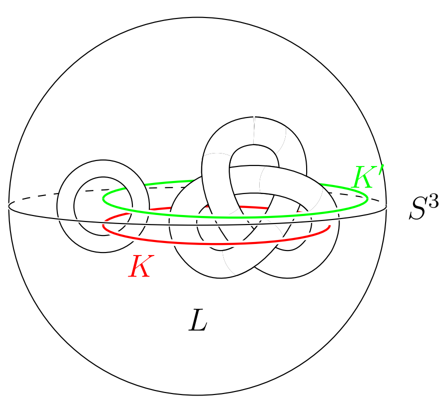
$$I(S^1 \times S^2) = \eta \Big\langle$$


$$\Big\rangle = \eta \cdot \eta \cdot \langle \omega \rangle_0 = 1$$

since $\sigma(L) = 0$ as the linking matrix is just (0). The above computation is "correct" in the sense that $I(S^1 \times S^2)$ must be $\dim V(S^2)$ (and therefore an integer) in TQFT. Recall that a TQFT gives dimension of the vector space M^2 for product of S^1 and M^2.

Remark 13.24. Like with the Jones polynomial that are two normalization of $\langle \omega L \rangle$ that are useful. One where an unknot gets 1 and one where the empty link gets 1. In our case, we want S^3 to be normalized and other time we want $S^1 \times S^2$ to be normalized.

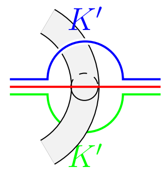
There is an extension of the above theorem giving us an invariant of framed links in M too. Suppose $M = S^3 - N(L) \cup_{L_i} (S^1 \times B^2)$ where we glue a solid torus into each link component L_i. Suppose we have a framed link inside M.



If K is a framed link in M, we can isotope K into $K' \subset S^3 - N(L)$ part of M (i.e., make it disjoint from the surgery tori) and compute

$$I(M, K) := \eta \kappa^{-\sigma(L)} \langle \Omega L \cup K' \rangle.$$

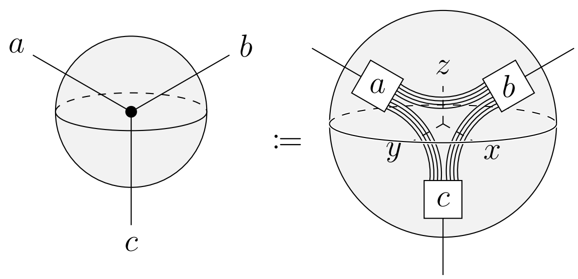
This is invariant because isotopy of K' corresponds to connect sum with a meridian of solid torus in M, or to connect sum to longitude of the corresponding torus in S^3, which is a handle slide move. However, handle slide which is identity after evaluation.



Remark 13.25. This is generalization of skein theory to invariants in other three manifolds. If we have skein triple of knots sitting inside the manifold, they would correspond to three version of K'. However, the invariants in terms of S^3.

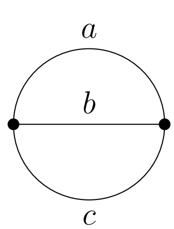
13.4. **Triads.** We have invariants of closed three manifolds and of framed links. We are getting towards constructing a TQFT. Before that we talk about triads.

Given a *triad* $a, b, c \in \mathbb{N}$ such that $a \leq b + c$ (and permuted version), $a + b + c$ is even, $0 \leq a, b, c \leq r - 1$, we can construct an element in $\mathcal{S}(B^3; a + b + c \quad \partial points)$ by joining the idempotents $f^{(a)}, f^{(b)}, f^{(c)}$ as follows.



Here x, y, z are "internal" number of strings that are used in connecting the idempotents as shown in the picture above. Here $x + y = c$, $x + z = b$ and $y + z = a$. Solving this equation, we can for example see that $x = \frac{b+c-a}{2}$.

We can construct trivalent graphs with labelled (with number) edges such that each trivalent vertex is interpreted as above and evaluate those as skein elements. For instance, we can now evaluate (after a nasty calculation)

$\Theta_{abc} :=$  $\in \mathcal{S}(S^3)$

$$= \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{x+z-1}!\Delta_{x+y-1}!}.$$

Here, $\Delta_n! := \Delta_n \Delta_{n-1} \ldots \Delta_0$.

The main observation is that $\Theta_{abc} \neq 0$ precisely when $0 \leq a, b, c \leq r-2$ and $a+b+c \leq 2(r-2)$. We call such traids *r-admissible.* In representation theory[11] this corresponds to the fact that

$$\mathrm{Inv}_{U_q\mathfrak{sl}(2)}(S^aV \otimes S^bV \otimes S^cV) \cong \begin{cases} \mathbb{C} & abc \text{ is admissible,} \\ 0 & \text{otherwise.} \end{cases}$$

Not at a root of unity the set of irreducible representation of the quantum group correspond to the set of irreducible representation of the classical group. For $SL(2)$, these are just indexed by the natural numbers. That's what a, b, c are doing in our context. They label the irreducible representation of the quantum group. Then the triad correspond to $V^a \otimes V^b \to V^c$ in the tensor product category. Equivalently, they correpond to the invariants in the tripple tensor product of irreducible representation. Here, a, b, c correspond to quantum symmetrizer. In the classical representation theory a symmetrizer is just the sum over $\frac{1}{a!}\sum Pictures$ where the sum runs over all the permutations of a strings. Idempotents are quantized version of.

For quantum groups, the set of irreducible representation truncate in some way in the sense that we get some modules that are good and some that are bad. The good ones are the ones in the range 0 up to $r-2$. The space of invariants also differ. The classical condition does not have $a+b+c \leq 2(r-2)$. This had to do with quantumness (at root of unity) of the quantum group.

13.5. **Reduced skein space.**

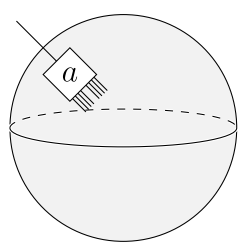
Definition 13.26. Corresponding to any skein space $\mathcal{S}(M, c)$ on a compact oriented 3-manifold with c points on ∂M, we have a *reduced skein space*

$$\mathcal{R}(M, c) = \frac{\mathcal{S}(M, c)}{f^{(r-1)} \equiv 0}.$$

The $f^{(r-1)} \equiv 0$ is new local relation to kill the highest idempotent.

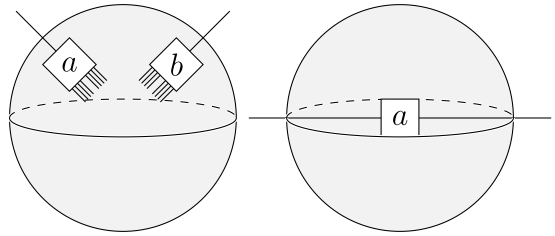
Remark 13.27. Handle slide identity is true in $\mathcal{S}$(genus 2 handlebody) only "mod terms with $f^{(r-1)}$" but it is identity in $\mathcal{R}$(genus 2 handle body).

Example 13.28. (1) $\mathcal{R}(S^3) = \mathbb{C}$ with basis vector $[\phi]$, the empty skein inside S^3.
(2) $\mathcal{R}(B^3) = \mathbb{C}$ with basis vector $[\phi]$.
(3) $\mathcal{R}(B^3, a)$ with a single idempotent $f^{(a)}$ attached at boundary is 0 except when $a = 0$ when its $\mathbb{C}$ as above.

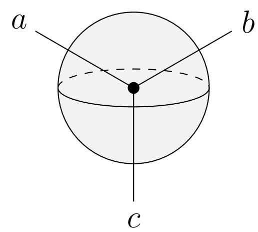


[11] If you are just doing combinatorics there is no guidance. We are secretly building representations of quantum groups at a root of unity.

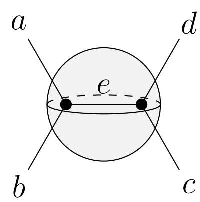
(4) $\mathcal{R}(B^3, a, b) = 0$ except when $a = b$ when its $\mathbb{C}$ generated by by elements that look like a string going through the ball (see picture on the right).



(5) $\mathcal{R}(B^3, a, b, c) = 0$ except when (a, b, c) is an admissible triple. It means that $a \leq b+c$ and cyclic permutations of that. $a+b+c$ is even, $0 \leq a, b, c \leq r-2$ and $a+b+c \leq 2(r-2)$. This condition is equivalent to $\Theta_{abc} \neq 0$.

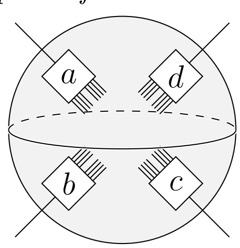


(6) The following lemma shows that

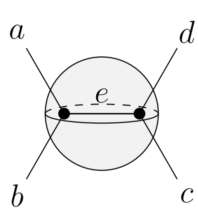
$\mathcal{R}(B^3, a, b, c, d) = \Big\langle$  $\Big\rangle$,

where abe and cde are admissible triads.

Lemma 13.29. *The reduced skein space of a ball B^3 with 4 boundary idempotents a, b, c, d*



has a basis of the form



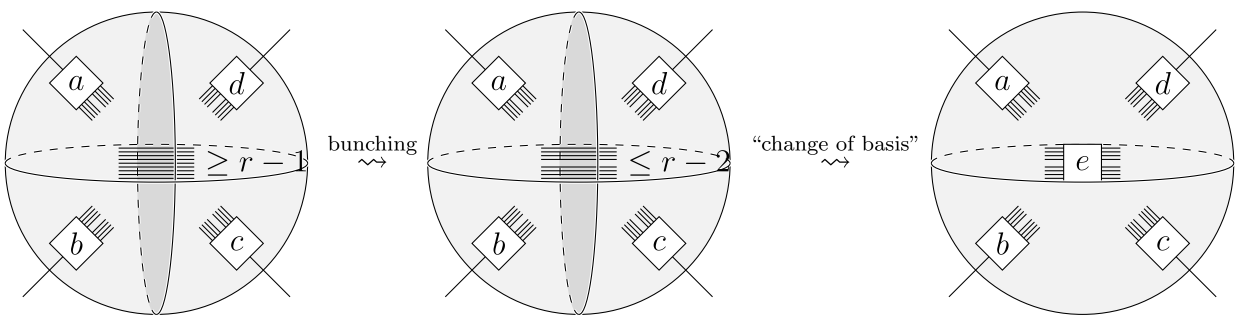
where abe and cde are admissible triples.

Proof. Use a "bunching" trick which can be used in many other circumstances.

First we can use the skein relations to get rid of all the crossings in our diagram. Further, if there is a backtrack, it gets annihilated by the idempotents. Therefore, we can get rid of the backtracks in the diagram. This will result in writing a given diagram as a linear combination of diagrams with certain number of strings crossing through a "middle disc" of B^3. Given an element with more than $r-1$ strings in a *bunch* crossing the middle disc, we can use

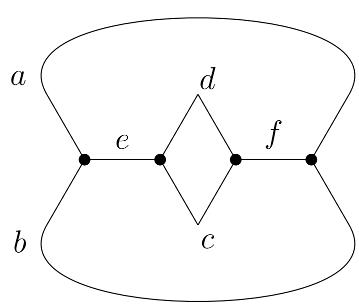
$$\mathbb{1}_{r-1} = f^{(r-1)} + (\text{backtracks}).$$

The first term vanishes in $\mathcal{R}$. Therefore, if we have $r-1$ strings in a bunch going through the middle disc, we can replace it with linear combination of diagrams with fewer strings crossing through the middle disc and with some backtracks which are annihilated by the idempotents. We can continue this process until we get at most $r-2$ strings passing through the middle disc. This allows us to write the original diagram in terms of linear combination of the "spanning set" consisting of diagrams that have at most $r-2$ strings in the middle disc.



Note that any collection of at most $r-2$ strings passing through the middle disc can be written in terms of idempotents $f^{(e)}$, $0 \leq r \leq r-2$, in the middle disc plus the ones with fewer strings and some backtracks. We can induct on the number of string crossing through the middle disc to get a different version of a spanning set. Since the diagram is closed, a, b, e and c, d, e have to be connected. Such triads exist only if these are admissible triples. This shows claimed basis spans.

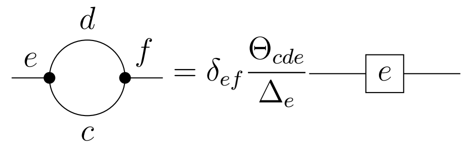
In fact, we claim that these elements are linearly independent. We can check this by evaluating "doubled" versions using the Lemma below.



$$= \delta_{ef} \frac{\Theta_{cde}}{\Delta_e} \Theta_{abe}.$$

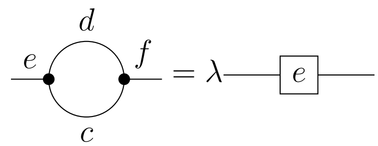
∎

Lemma 13.30 (Bubble relation)**.**

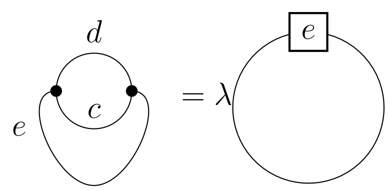


Proof. Note that when we evaluate (in B^3) the left hand side of the above equality, everything in the middle will have crossing less diagram. Further, there has to be a backtrack somewhere if we have idempotents e and f on two sides with different number of strings. In that case, the idempotent annihilates the backtrack.

If $e = f$ then we get



To compute λ we can close up the diagram on the both side to get,



However, the left hand side is Θ_{cde} and the right hand side is $\lambda \Delta_e$.

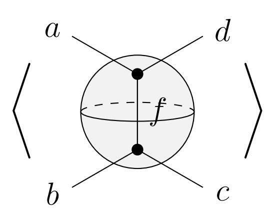
Really, we have shown that the claimed basis is actually orthogonal with respect to the Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathcal{R}(B^3, a+b+c+d)$ such that

$$\langle x, y \rangle$$

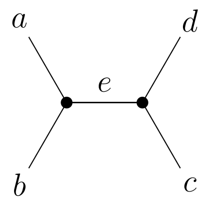
is obtained by gluing $\overline{x}$ and y. Here $\overline{x}$ is the mirror image of x by switching A and $\overline{A}$. $\square$

Fact 13.31. Lens space is not named after Mr Lens. There is no Mr. Lens. Skein space is not named after Mr. Skein. There is no Mr. Skein.

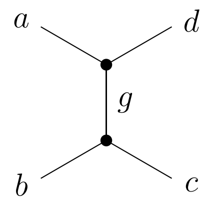
There seems to be an asymmetry in the above Lemma in our choice to connect a with b and c with d. There is an alternative basis that connects a with c and b with d

$\mathcal{R}(B^3, a, b, c, d) =$ 

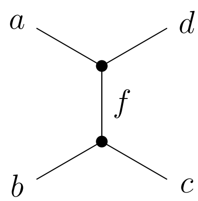
where adf and bcf are admissible. These two basis are related by a change of basis matix (called "quantum $6j$ symbols in physics literature. j represents spin and they write $j_1, \ldots, j_6$.):



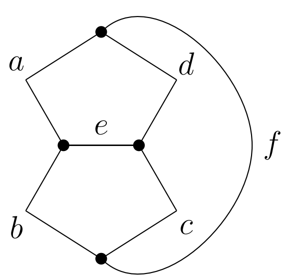
$$= \sum_g \begin{Bmatrix} a & d & e \\ b & c & g \end{Bmatrix}$$



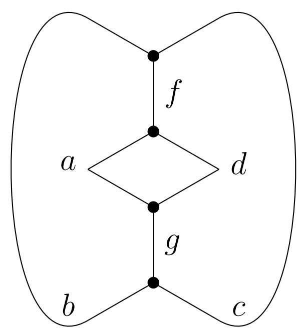
The $6j$ symbol $\begin{Bmatrix} a & d & e \\ b & c & g \end{Bmatrix}$ denotes a matrix with indices e and g that also depend on a, b, c, d. We can calculate the $6j$ symbols by pairing the whole equation with the element



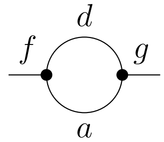
on both side to get



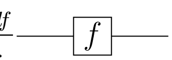
$$= \sum_g \begin{Bmatrix} a & d & e \\ b & c & g \end{Bmatrix}$$



But we know a bubble relation



$$= \delta_{fg} \frac{\Theta_{adf}}{\Delta_f}$$

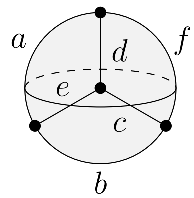


So the right hand side is

$$\begin{Bmatrix} a & d & e \\ b & c & f \end{Bmatrix} \frac{\Theta_{adf}}{\Delta_f} \frac{\Theta_{cbf}}{\Delta_f} \Delta_f.$$

So the $6j$ symbol is equivalent to the tetrahedral diagram

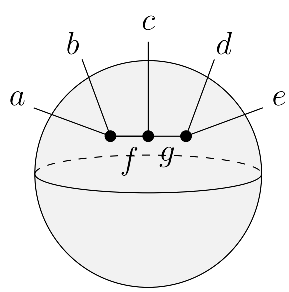
$$\frac{\Delta_f}{\Theta_{adf}}\frac{\Delta_f}{\Theta_{cbf}}\frac{1}{\Delta_f}$$



Note that this tetrahedral graph does have an interesting (tetrahedral symmetry). This matches the fact that there is secret tetrahedral symmetry of the classical $6j$ in physics.Namely, we can label the edges of a tetrahedron by the 6 numbers and permute them via the symmetry of tetrahedron.

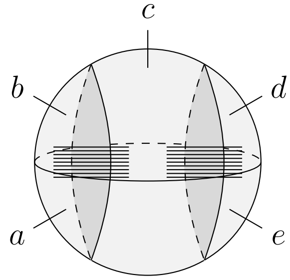
Remark 13.32. The change of basis via the $6j$ matrix is called recoupling of a, b, c, d in the physics literature.

Example 13.33. In general, we can similarly compute $\mathcal{R}(B^3, \text{lots of idempotents on } \partial)$ by choosing a trivalent graph spanning the boundary points and using all admissible labellings of this graph to define a basis. For instance, the basis of $\mathcal{R}(B^3, a, b, c, d, e)$ contains elements of the form



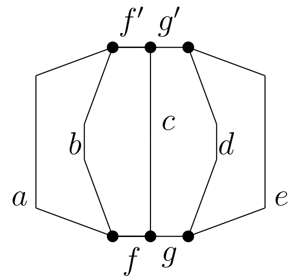
where f and g vary such that (abf), (cfg) and dge are all admissible. To check these form a basis, we first use the bunching idea to show these spans. Moreover, by pairing with similar diagrams on the outside, we can show linear independence.

More precisely, we look at strings crossing through "dual discs" to the internal edges,[12] to reduce to the case where at most $f, g \leq r-2$ strings go through the dual discs as shown below.



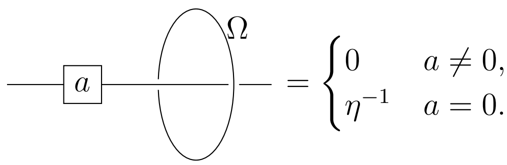
Then we use use the bubble identity to evaluate each element in the spanning set with its mirror. This gives us a hermitian inner product that makes the spanning set an orthogonal basis.

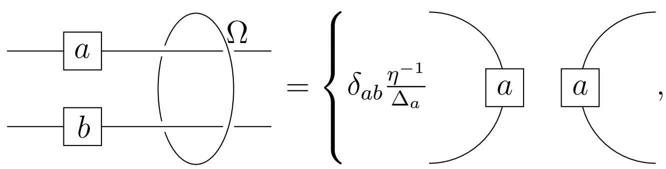
[12]If we have n boundary points then there will be $n-3$ internal edges in the resulting trivalent graph

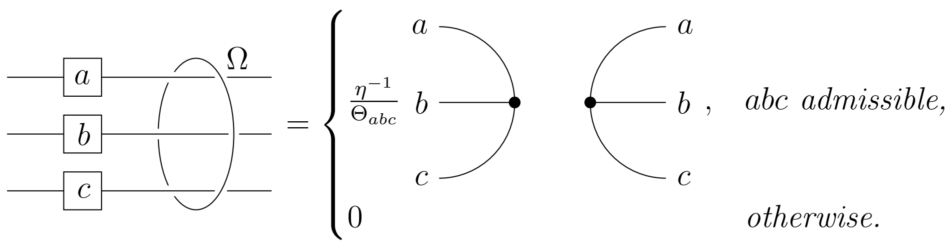


This following is a version of Lemma 13.19. They are useful in calculations to know that in $\mathcal{R}(B^3, \partial points)$. In fact, they show that closed Ω annihilates strings passing through it.

Lemma 13.34 (Fusion identities)**.** *The following identities hold*

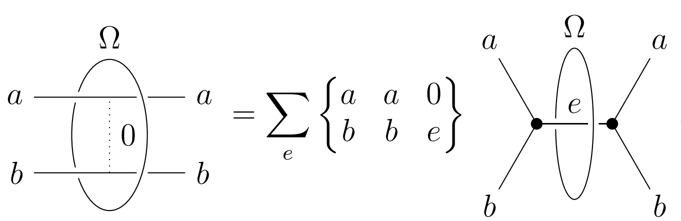






The first one is called the one string fusion identity and the second one is called the two string fusion identity.

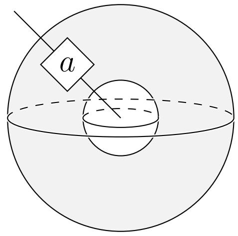
Proof. The first identity follows the same proof as in Lemma 13.19. The second statement follows by using the change of basis via $6j$ matrix by thinking of the given diagram as follows:



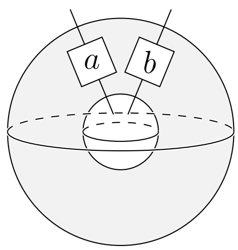
Then we can use the first string fusion identity on the right hand side and the definition of the $6j$ matrices elements to get the desired identity. $\square$

These lead to further identities in $\mathcal{R}(S^2 \times I, \partial \text{ points})$.

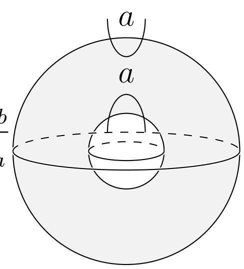
Lemma 13.35.



$$= \begin{cases} 0 & a \neq 0, \end{cases}$$

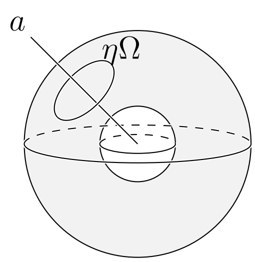


$$= \frac{\delta_{a,b}}{\Delta_a}$$



.

Proof. Note that the first fusion identity implies that

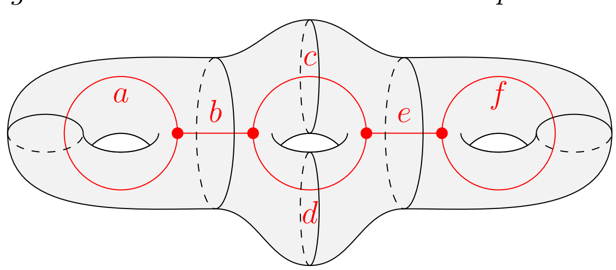


$$= 0$$

except when $a = 0$. On the other hand we can isotope $\eta\Omega$ to the other side of S^1 and cancel it via $\langle \eta\Omega \rangle_0 = 1$ to get the given picture.

We can use a similar trick to prove the second identity. ☐

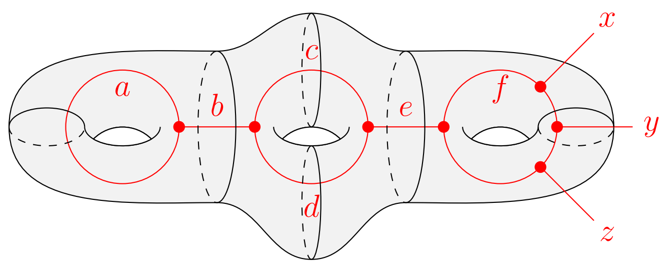
Lemma 13.36. $\mathcal{R}$(*handlebody of genus* g) *has basis given by all admissible labellings of any trivalent graph drawn dual to a system of* $3g - 3$ *"cutting discs" (cutting it into solid pairs of pants). For instance, if* $g = 3$ *then the basis elements correspond to the pictures of the form:*



Proof. We use bunching (along cutting discs) to prove that such elements span $\mathcal{R}$. To prove linear independence, we glue two such things together to make a connect sum $\#^g S^1 \times S^2$ (double of a handlebody). ☐

Remark 13.37. (1) We can count the dimension of $\mathcal{R}$(handlebody of genus g) but they are not particularly nice.

(2) For a handlebody with boundary points, similar results hold. For instance, when $g = 3$ and the boundary labeling are x, y, z then basis elements are of the form:

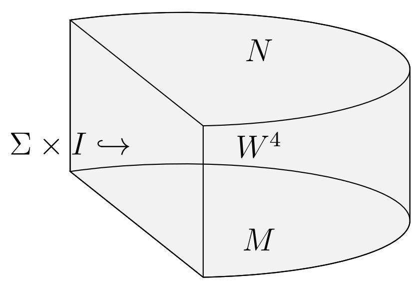


Theorem 13.38. *The reduced skein space $\mathcal{R}(M^3, c)$ depends only on $(\partial M^3, c)$ where c is labelling of the boundary. In particular, if M^3 and N^3 are manifold with same boundary (Σ, c) then*

$$\mathcal{R}(M, c) \cong \mathcal{R}(N, c).$$

Remark 13.39. Since we have a classification of genus g surfaces Σ_g, the skein space $\mathcal{R}(M, c)$ of a manifold M^3 with boundary Σ_g is isomorphic to $\mathcal{R}$(handle body of genus g, c) for which we have outlined the computation above.

Skectch of the proof of Theorem 13.38. Take a relative 4-cobordism W^4 between the two three manifolds M^3 and N^3 which can be schematically representated as follows.[13]



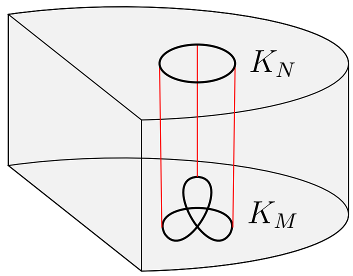
We can run a Morse theory argument (relative to the boundary so that the Morse function is the height function on the side $\Sigma \times I \to \mathbb{R}$) on W^4 to get a handle decomposition. The height function on the boundary guarantees that there are no critical points on $\Sigma \times I$ so that handles are attached on the interior.

We can assume that W is connected even if M and N are not connected. Recall that as we go from bottom to the top, attaching 0-handle is a birth of a S^3 bounded by B^4. By altering our Morse function we avoid the birth of S^3 since we assume W to be connected which guarantees that there is a 1-handle that lands on this S^3. Therefore, we can cancel 1-handle with the 0 handle just like in surface case. We can reverse the diagram and get rid of 4 handles as well.

So we must analyze what happens to the reduceed skein space when attaching $1, 2, 3$-handles to a 3-manifold.

[13]The existence of such W^4 is a special case of the fact that every closed oriented 3 manifold bounds a four manifold because $\Omega_3^{SO} = 0$.

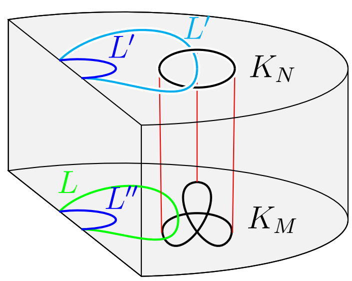
Case of 2-handles: Recall that a 2-handle is $B^2 \times B^2$ (is a 4-ball) so that $\partial(B^2 \times B^2) = S^1 \times B^2 \cup_{S^1 \times S^1} B^2 \times S^1$ where the two solid tori are interlocked to form S^3.[14] Therefore, a 2-handle attaches via $S^1 \times B^2 \subset M$, which is just a thickened knot $K_M \subset M$ and leaves a solid torus $K_N \subset N$ at the top.



In particular, $M - K_M \cong N - K_N$ (in the sense of neighborhood).

Define a map

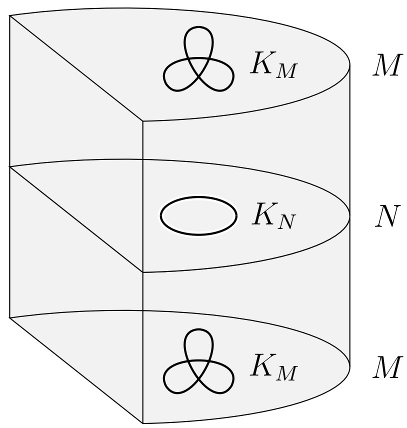
$$\begin{aligned} \mathcal{R}(M, c) &\to \mathcal{R}(N, c) \\ [L] &\mapsto [L' \cup \Omega K_N], \end{aligned}$$



To get L', first isotope L in M disjoint from K_M to get L''. Then transfer L'' to L' in $N - K_N$ via diffeomorphism $M - K_M \cong N - K_N$. Then add in $\mathcal{R}_N$ the element "Ω" on K_N. Note we can isotope L on either side of K_M. This gives in L'' that differ from each other up to a connect sum with the meridian of K_M. On top it corresponds to whether L' goes around the longitude of K_N. More precisely, L''s (L' and L') differ by handle slide over K_N. Adding ΩK_N cancels out this effect by the handle slide property of Ω. In particular, the map is well defined of entire reduced skein space. This map also preserves the skein relations.

We claim that this map is actually an isomorphism. Think about the composite with the reverse bordism and the corresponding map $\mathcal{R}(M, c) \to \mathcal{R}(N, c) \to \mathcal{R}(M, c)$.

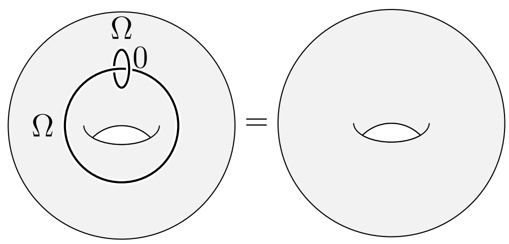
[14] This is the standard Heegard splitting of S^3.



The effect of this will be

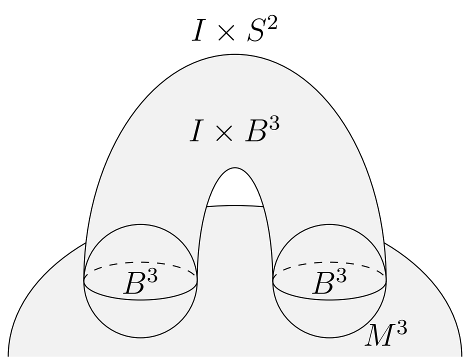
$$[L] \mapsto [L' \cup \Omega K_N] \mapsto [L'' \cup \Omega K_M^* \cup \Omega K_M],$$

where K_M^* is the 0-framed meridian of K_M. One that ΩK_N is the longitude, so under the above map it gets mapped to ΩK_M^*, the meridian. On the other hand, ΩK_M is the longitude of K_M. On the other hand,



in $\mathcal{R}(S^1 \times B^2)$. In fact, we can use the first fusion identity in Lemma (13.34) to get $\langle \Omega \rangle_0 = \eta^{-1}$ and $\Omega = \eta\omega$. Therefore, $[L'' \cup \Omega K_M^* \cup \Omega K_M] = [L''] = [L]$.

Case of 1-handle also uses similar idea. In fact, 1-handle is $B^1 \times B^3$ with boundary $\partial(B^1 \times B^3) = (S^0 \times B^2) \cup_{S^0 \times S^2} (I \times S^2)$. Attaching 1-handle amounts to removing two B^3's in M and replacing it by $B^1 \times S^2$.



Then we define a map

$$\begin{aligned} \mathcal{R}(M, c) &\cong \mathcal{R}(M - 2B^3, c) \quad \text{(via isotopy)} \\ &\cong \mathcal{R}(N - S^2 \times I, c) \quad \text{(via transferring map as in2handle case)} \end{aligned}$$

$$\to \mathcal{R}(N, c) \quad \text{(close up the hole)}.$$

In fact, this last mapping is an isomorphism. This is because of Lemma 13.35. In fact, if N contains $S^2 \times I$, then we can assume that skein elements in $\mathcal{R}(N, c)$ are disjoint from $S^2 \times I$. Otherwise, we can use bunching argument in $S^2 \times I$ so that there is an idempotent a passing through $S^2 \times I$. Unless the idempotent $a = 0$ (i.e., the skein is disjoint from $S^2 \times I$), Lemma 13.35 that the diagram vanishes. Therefore, the reverse of the mapping makes sense. And can be used to define what 3-handle does. □

In particular, for any closed 3-manifold M^3,

$$\mathcal{R}(M) \cong \mathcal{R}(S^3) = \mathbb{C}.$$

However, the first isomorphism is not canonical. To make is canonical, we need to choose a bordism as in the proof of the above theorem.

For instance, we view M is $S^3 \cup$2-handles[4] i.e., M is obtained via a surgery on a framed link $K \subset S^3$.[15] Then if we use the proof of the Theorem, we get a map

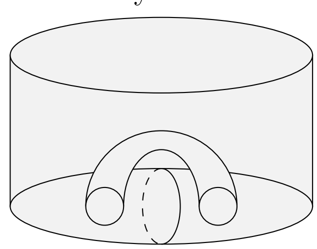
$$\mathcal{R}(M) \to \mathcal{R}(S^3) = \mathbb{C}$$
$$[L] \mapsto [L \cup \Omega K] \in \mathbb{C}.$$

Note that $[L \cup \Omega K]$ is the invariant $I(M, L)$ of the manifold link pair (M, L) as defined earlier except without the noramlizing factor $\eta\kappa^{-\sigma(K)}$. In particular, this is a mapping that evaluates the invariants of (M, L).

Hence, observe that the mapping

$$\mathcal{R}(M) \to \mathbb{C}$$

depends on the 4 bordism W^4 chosen only in a weak sense (via its σ).



The following theorem shows the "weak" functoriality of $\mathcal{R}$ at the level of 3 manifolds and 4 manifolds. We already saw that for given M^3 and N^3, we can find morphisms between $\mathcal{R}(M)$ and $\mathcal{R}(N)$ that depends only weakly on W^4.

Theorem 13.40. *There exists a $3+1$ TQFT R but in a weak form*[16] *For closed 3 manifold M^3, define $R(M) := \mathcal{R}(M) \cong \mathbb{C}$. For 4 manifold bordism W^4, we break it up into handles and define morphism for handle attachments as follows.*

(1) For 0 handle, we set

$$\mathcal{R}(M) \to \mathcal{R}(M \sqcup S^3) = \mathcal{R}(M) \otimes \mathcal{R}(S^3)$$
$$x \mapsto x \otimes \eta^{-1}[\emptyset].$$

[15] Lickorish's theorem says that every M^3 can be obtained from surgery on a framed link.
[16] This is isomorphic to something called "Crane–Yetter" theory.

(2) *For* 1*-handle, we set*

$$[\mathcal{R}(M) \to \mathcal{R}(M - 2B^3) \xrightarrow{\cong} \mathcal{R}(N - S^2 \times I) \xrightarrow{\cong} \mathcal{R}(N)] \times \eta.$$

(3) *For* 2*-handle, we set*

$$\mathcal{R}(M) \to \mathcal{R}(N)$$
$$[L] \mapsto [L \cup \Omega K_N].$$

(4) *For* 3 *handle inverse the map for* 1*-handle map times* η^{-1}.

(5) *For* 4 *handle, we set*

$$[\mathcal{R}(S^3) \to \mathbb{C}] \times \eta$$

where $[\mathcal{R}(S^3) \to \mathbb{C}]$ *is the standard evaluation.*

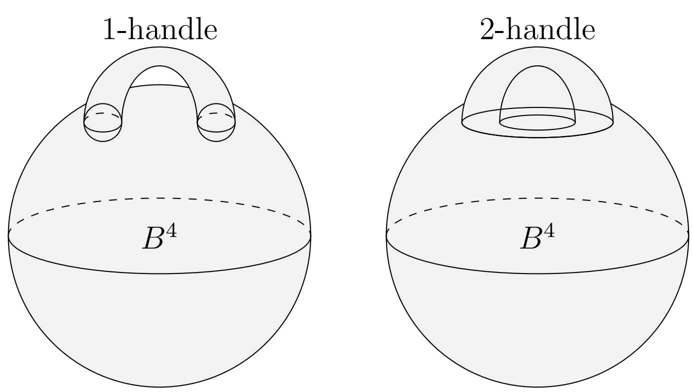
Remark 13.41. We claim that R is a TQFT. Firstly, mappings are well defined and in fact independent of the handle decomposition. (To prove this, we need to use "Cerf theory," which is Morse theory for one parameter family of Morse functions. This gives us an understanding of two handle decomposition in terms of handle moves (isotopy of handles). For instance, we can assume that first 0 handles are attached then 1 and so on just like how we build CW complexes. Further, we must understand what happens when we isotope two handles of same index. For instance, two handles of same index might slight over each other. Further, we must analyze the effect of handle slides and birth/death of canceling pairs of handles.)

Theorem 13.42. *If* W^4 *is a closed manifold then we can show that* $R(W^4) = \kappa^{\sigma(W)}$.

Proof Sketch. The reason is effectively that the bordisms

$$S^1 \times B^3, B^2 \times S^2 : \emptyset \to S^1 \times S^2$$

correspond to attaching a 1-handle or a 2-handle (onto a unkotted torus) on B^4. Note that both bordims have boundary $S^1 \times S^2$.)



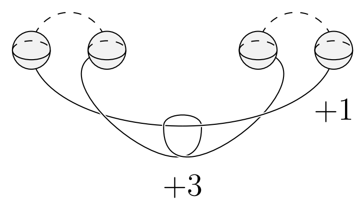
These two bordisms define the same mapping

$$\mathbb{C} \to \mathcal{R}(S^1 \times S^2),$$
$$1 \mapsto [\emptyset] \in \mathcal{R}(S^1 \times S^2).$$

and this causes a degeneration of complexity.

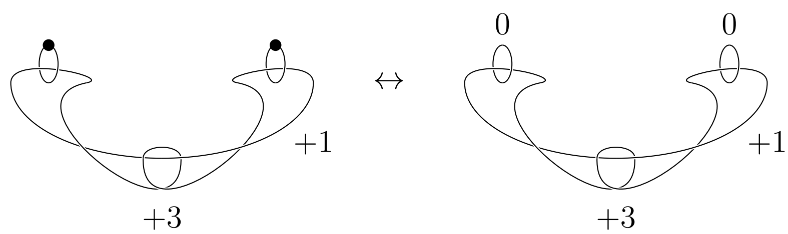
If we can't distinguish 1 handle from 2 handle, we lose a lot of information about W^4.

In particular, if we start with a "handle diagram" of W of the form with paired 3 balls. A handle decomposition of a connected 4 manifold starts with a 4-ball. We attach 1 handles to it and then 2-handles and so on. Handle diagram is a picture of handle decomposition in $S^3 = \partial B^4$. Solid balls are B^4. Dashed lines represent 1-handles. Since attaching 2 handle is surgery on a framed link, we draw solid lines with numbers that encode framing to represent the surgery framed link.



It is a fact that we don't need to draw 3 and 4 handles. There is a theorem that says that any union of 3 and 4 handles is a solid 4-dimensional handle body which looks like upside down picture of B^4 with a 1-handle. The upshot is that once we have where prescribed 1 and 2 handles go on B^4, there is a particular way in which 3 and 4 handles need to get attached to get closed manifold W^4. In any case, the result of doing surgery up to 2 handle is a connect sum of $S^1 \times S^2$ because the result has to be the boundary of the solid handlebody that we will cap off with 3 and 4 handle.

We can also draw handlebody diagrams using Freedman's notation. Namely, we replace the dotted 1 handles on a pair of 3 ball with an unknot with a dot on it. The fact that we can't distinguish 1 handle from 2 handle means that that dotted unknot is actually a 0 framed unknot.



The result is a framed link giving a surgery presentation of a connect sum of $S^1 \times S^2$ and hence an invariant which is of the form

$$\eta^{power} \times \kappa^{power}.$$

If we work out the values of the power, it turns out to be $\sigma(W)$ after normalization. $\square$

14. 2+1 TQFT revisited

Our final goal is to show how we can derive Witten–Reshetikhin–Turaev $2+1$-TQFT (WRT) out of the $3+1$ TQFT that we built using the reduced skein spaces. This part of the notes will be little technical. There is an alternative way to describe WRT using very gritty, gnarly, down to earth methods.

Recall that for any (M^3, c) with framed points in its boundary, we defined its reduced skein space $\mathcal{R}(M, c)$. Further, a relative cobordism $W^4 : M \to N$ induces an isomorphism of reduced skein spaces $\mathcal{R}(M, c) \cong \mathcal{R}(N, c)$, so we claimed that $\mathcal{R}(M, c)$ depdends only on the boundary (Σ, c).

Moreover, we defined $3+1$ TQFT R that assigns

(1) for closed M^3, a one dimensional vector space $R(M) = \mathcal{R}(M) \cong \mathbb{C}$.
(2) For every bordism $W^4 : M \to N$, a morphism$R(W) : R(M) \to R(N)$ defined using handle attachment formula.

In particular, if W is closed then $R(W^4) = \kappa^{\sigma(W)}$.

Theorem 14.1 (Combination theorem)**.** *In the relative case, the isomorphism $\mathcal{R}(M, c) \to \mathcal{R}(N, c)$ depends only on W via its signature.*

Namely, if two W^4 relative cobordisms have same signature then they induce equal isomorphism.

Therefore, we can "almost" do a sort of dimensional reduction to a $2+1$ theory by associating $\mathcal{R}(M, c)$ to (Σ, c). However, we need an extra structure to control the "signature ambiguity" that comes from different bordisms W between M and N that makes the isomorphism $\mathcal{R}(M, c) \cong \mathcal{R}(N, c)$ non-canonical. The goal is to get rid of the effect of signature of W^4 by requiring more (p_1) structures on our $2+1$ cobordism category. A *p_1-structure* is akin to an orientation, spin structure or framing of a manifold as follows. We will use the homotopy theoretic description of the p_1 strucutre.

14.1. p_1 **structures.** Given a manifold M, we have a classifying map for its (stable) tangent bundle

$$\begin{array}{ccc} TM \cong f^*(T) & \longrightarrow & T \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BO. \end{array}$$

Here, BO, the classifying space is the base of the universal bundle T.

There is a fibration sequence

$$K(\mathbb{Z}_2, 0) \to BSO \to BO \xrightarrow{w_1} K(\mathbb{Z}_2, 1).$$

Here, w_1 is the first Steifel–Whitney class thought of in a homotopy set theoretic sense. Normally, w_1 is the first cohomology class on BO with coefficients in $\mathbb{Z}_2$. However, $H^1(BO; \mathbb{Z}_2)$ is the same as $[BO, K(\mathbb{Z}_2, 1)]$, the homotopy classes of maps from BO to Eilenberg–Maclane spaces $K(\mathbb{Z}_2, 1)$. The homotopy fiber of w_1 is BSO. A way to see this is applying the classifying map functor to the inclusion $SO \hookrightarrow O$. Moreover, BSO has a homotopy fiber $K(\mathbb{Z}_2, 0)$.

Definition 14.2. A *"pointwise" orientation* on M is a choice of lift

$$\begin{tikzcd}
 & K(\mathbb{Z}_2,0) \arrow[d] \\
 & BSO \arrow[d] \\
M \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & BO \arrow[d, "w_1"] \\
 & K(\mathbb{Z}_2,1)
\end{tikzcd}$$

Remark 14.3.

(1) Such a lift exists if and only if $w_1(TM) = 0$ which is true if and only if M is orientable. If an orientation exists then the homotopy classes of lifts are classified "affinely" by $[M, K(\mathbb{Z}_2, 0)] = H^0(M; \mathbb{Z}_2)$. In particular, if $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts then $\tilde{f}_1 - \tilde{f}_2$ defines a map $M \to K(\mathbb{Z}_2, 0)$. These are functions from path components of M to $\mathbb{Z}_2$. This agrees with the fact that two orientations differ by functions that says whether they agree/disagree on each component.

(2) Talking about relative orientation and gluing that happensin cobordisms. If M has boundary then we can think of orientation on the boundary as a particular lift of f to $\tilde{f}$ over the boundary. Associated with this lifting, there is a relative lifting problem. Can we extend the lift on the boundary to all of M? If we can do so, we say the manifold is orientable relative to the orientation on the boundary.

Definition 14.4. A *spin structure* on an oriented manifold M with orientation f is a choice of lift:

$$\begin{tikzcd}
 & K(\mathbb{Z}_2,1) \arrow[d] \\
 & BSpin \arrow[d] \\
M \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & BSO \arrow[d, "w_1"] \\
 & K(\mathbb{Z}_2,2).
\end{tikzcd}$$

Remark 14.5. $BSpin$ is the double cover of BSO. Such a lift exists if and only if $w_2(TM) = 0$ in $H^2(M; \mathbb{Z}_2)$ which is true if and only if M is spin. Further, different litfs are measured affinely by $[M, K(\mathbb{Z}_2, 1)] = H^1(M; \mathbb{Z}_2)$. This is akin to the fact that different spin structures are affinely classified by $H^1(M; \mathbb{Z}_2)$.

Recall that first Pontrjagin class $p_1 \in H^4(M; \mathbb{Z})$ (for real bundle E, $p_1(E) \coloneqq c_2(E \otimes \mathbb{C})$) can be represented by the differential form "$\mathrm{tr}(F \wedge F)$" with some normalization factor. Similar to what we did with the cohomology classes w_1 and w_2, we can define p_1 structure using the first Pontrjagin class.

Definition 14.6. A p_1 *structure* on an oriented manifold with orientation f is a lift to the homotopy fiber F of p_1 :

$$\begin{tikzcd} & K(\mathbb{Z},3) \arrow[d] \\ & F \arrow[d] \\ M \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & BSO \arrow[d, "p_1"] \\ & K(\mathbb{Z},4). \end{tikzcd}$$

Remark 14.7.

(1) A p_1 structure exists on M if and only if f $p_1(TM) \in H^4(M;\mathbb{Z}) = 0$. Further, the different lifts are measured affinely by $H^3(M;\mathbb{Z})$.
(2) The moral p_1 structure is a bit like picking a framing on a low dimensional manifold, except we "bypass spin structure part." By this we mean the following. If we are trivializing tangent bundle i.e., pick a framing on it, we first choose an orientation. Then we choose a spin structure on it. Then the next cohomology class that we want vanish would be p_1. For instance, the existence of a trivialization of the tangent bundle of $M^{\leq 4}$ would be the same as requiring $p_1(TM) = 0$. However, if we bypass the spin case, it makes it easier to find p_1 structures easier.

Fact 14.8.

(1) A p_1 structure is a "pointwise" object, but we will always end up considering them up to homotopy. For example, the category $\widetilde{\mathbf{Cob}}_{2+1}$ could be defined to have the following data.
    (a) Objects: surfaces with a (pointwise) p_1 structure.
    (b) Morphisms: 3 manifold bordisms with p_1 structure considered up to homotopy relative to ∂.

    Thus, we can glue we can glue "p_1 bordisms" in the obvious way.
(2) All p_1 structures on a surfaces are actually homotopic (because $H^3(\Sigma;\mathbb{Z}) = 0$.)
(3) On a 3 manifold with boundary, all p_1 structures are classified by $H^3(M, \partial M;\mathbb{Z})$ ($= \mathbb{Z}$ for a connected manifold. In particular, we have $\mathbb{Z}$-indexed family of M^3 with p_1 structures.)
(4) To "measure" the integer difference between p_1 structures, we use Hirzebruch's signature theorem as follows. Given two different p_1 structures (M,α) and (M,β) agreeing on ∂M, we can form $\overline{M} \cup M$ with $\overline{\alpha} \cup \beta$. Then take a connected 4 manifold W with $\partial W = \overline{M} \cup M$ (we can find such W because $\Omega_3^{SO} = 0$) and define

$$\sigma(\beta - \alpha) = p_1(W, \overline{\alpha} \cup \beta) - 3\sigma(W).$$

    Here, p_1 is the relative p_1-structure in $H^4(W, \partial W;\mathbb{Z}) = \mathbb{Z}$. It is the obstruction to extending the p_1 structure on the boundary over the interior of W^4.

    It turns out that $\sigma(\beta - \alpha)$ is independent of the choice of W because if W and W' are chosen, we can glue them together to get a closed manifold $W \cup W'$, where by

(Hirzebruch's signature theorem)

$$p_1(W \cup \overline{W}') = 3\sigma(W \cup \overline{W}').$$

This shows that the formula for $\sigma(\beta - \alpha)$ is independent of W.

(5) A 4 cobordism "propagates" a p_1 structure from M to N by requirement that $p_1(\overline{\alpha}_M \cup \alpha_N) = 0$.

14.2. **WRT functor.**

Claim 14.8.1. *Suppose (M, α_M) and (N, α_N) are p_1-3manifolds with same boundary $(\Sigma, \alpha_\Sigma, c)$, we claim that there exists a canonical map*

$$\mathcal{R}(M, c) \cong \mathcal{R}(N, c)$$

given by picking any relative 4 cobordism and defining the isomorphism to be

$$\kappa^{-\frac{1}{3}p_1(W, \overline{\alpha}_M \cup \alpha_N)} \cdot R(W).$$

Proof. The factor $\kappa^{-\frac{1}{3}p_1(W, \overline{\alpha}_M \cup \alpha_N)}$ counterbalances the dependence of $R(W)$ on signature of W. Note that relative p_1 measures the signature of W. □

We now define the WRT theory.

Definition 14.9. The category $\widetilde{\mathbf{Cob}}'_{2+1}$ consists of the following data.

(1) Objects: p_1 closed surfaces with framed points.
(2) Morphisms: p_1 3-manifolds with framed links.

The $'$ indicates "relative" theory with framed points on Σ and links inside the manifold M that Σ bounds.

We can define $WRT(\Sigma, \alpha_\Sigma, c) := \mathcal{R}(M, c)$ for any 3-manifold M that bounds $\partial M = \Sigma$ in a sense. Technically, we can take the limit of the diagram sending

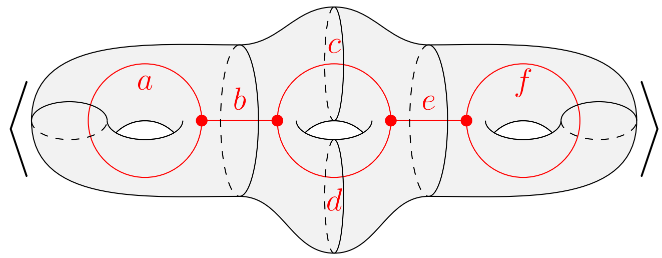
$$\begin{tikzcd} (M, \alpha_M) \arrow[r] \arrow[d, Rightarrow, "W"'] & \mathcal{R}(M, c) \arrow[d, "\kappa^{-\frac{1}{3}p_1} R(W)"] \\ (N, \alpha_N) \arrow[r] & \mathcal{R}(N, c) \end{tikzcd}$$

This limit effectively identifies all different $\mathcal{R}(M)$'s in a consistent way. In particular, the isomorphism $\mathcal{R}(M, c) \cong \mathcal{R}(N, c)$ is independent of W and its signature.

In practice, we simply pick one "model" and work with it. For example, given $(\Sigma_g, \alpha_\Sigma)$, the easiest 3-manifold to take is (H_g, α_H), the solid handle body of genus g where α_H extends α_Σ. This tells us that

$$\begin{aligned} &WRT(\Sigma_g, \alpha_\Sigma) \\ &= \langle \text{trivalent graphs (with admissible labels) dual to pair of pant decomposition of } \Sigma_g \rangle. \end{aligned}$$

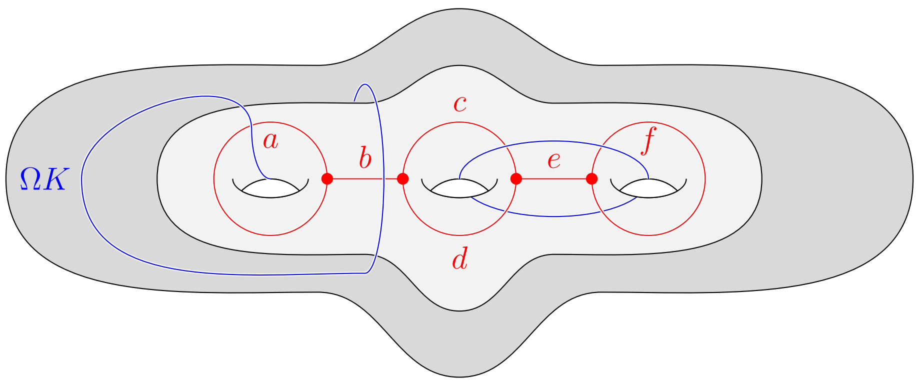
For instance,

$WRT(\Sigma_3, \alpha_\Sigma) =$ 

where the labels are admissible triads.

If we were given now a bordism $\Sigma_g \xrightarrow{M} \Sigma_g$, we would typically express it as surgery on $\Sigma \times I : \Sigma_g \to \Sigma_g$ with surgery on some link K and add these surgery curves, labeled by Ω, over input skein elements in $\mathcal{R}(\Sigma_g)$.

For instance, blue curve in the picture below is a surgery link K. Then the morpshism $\Sigma_3 \xrightarrow{M} \Sigma_3$ sends (abcdef) to $abcdef \cup \Omega K$ as pictured below of $\Sigma_3 \times I$ (a thickened H_3).



14.3. **What's next?** A next step for these notes would be a TQFT treatment of Khovanov homology, which is the categorification of the Kauffman bracket theory.

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