Algebraic Topology I

Based on lectures by Prof. Jeremy Hahn

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Last updated: March 7, 2023

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Preface

These notes grew out as transcription of lectures on algebraic topology (18.905) given by Prof. Jeremy Hahn in Fall 2021 at MIT. I have partly organized, cut down the repeated materials during lectures, and partly tweaked the presentation style, but have done nothing else.

The prerequisite for this course was algebra and topology (18.701 and 18.901 at MIT).

These notes do not cover fundamental groups and covering spaces. Instead, we cover homology, cohomology, and Poincaré duality, strictly following the first three chapters of [Mil16], the main reference for the course. In fact, the details omitted in these notes can be found in [Mil16].

The main goal of these notes is to come up with algebraic tools to distinguish topological spaces, focusing more on the algebraic and combinatorial aspects of the tools. For a geometric flavor, one can use [Hat05]. However, we have made some remarks on the correspondence between the algebraic and topological objects whenever possible. A summary of the correspondence is as follows:

	Topological objects	Algebraic objects
Models	Topological space, CW complex	Chain complex
Maps	Homeomorphism, Homotopy	Chain map, Chain homotopy
Constructions	Gluing, products, unions	Quotients, tensor products, direct sum
Category	Homotopy category	Derived category
Invariants	(Co)holes	(Co)homology

Table 1: Correspondence between algebraic and geometric objects

In any case, the main philosophy is to probe topological spaces using functions from model spaces (n-simplices) and form invariants using the class of functions. For the matter of computation, we break up spaces into model spaces (giving rise to CW structures) and compute the holes of the whole using the invariants of the pieces. The technicality comes in proving that the holes don't depend on the choice of the pieces.

Chapter 1 develops the theory of homology. Chapter 2 consists of some preliminary computational tools. Since the space of functions from model spaces is quite large from a computational perspective, we get around by giving an equivalent but computationally easier definition of homology in Chapter 3. Then in Chapter 4, we will work with the generalization of homology. In Chapter 5, we will define cohomology, the dual of homology group. It turns out that cohomology groups have a ring structure, making them more powerful invariants than homology groups. In Chapter 6, we will give an application of algebraic topology to geometry. Finally, in Appendix, we record some future directions that one can take after reading these notes.

Apology: I apologize for failing to provide extensive bibliography. Please see [Mil16] for more details on history and bibliography. I also apologize for the mistakes (which are due to the writer and not the instructor). To warn you of a logical hole, possible mistakes, or the fact that some paragraphs are not well written, I have included a road sign:

Please feel free to email me at rkoirala(at)ucsd(dot)edu if you have any comments or want to report mistakes.

Acknowledgements: I am grateful to Jeremy for teaching a wonderful class. I did not quite appreciate the algebraic approach while I was taking the class. However, over time, I have started to fall in love with the algebraic road for its elegance (even if it is less “intuitive” at times). I needed time to see that the algebraic objects are just the models of topological objects.

I also thank my family for their co-love and co-support.

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Notation

Ab	Category of abelian groups
Cat	Category of small categories
chAb	Category of chain complexes
$\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})$	Category of chain complexes of $\mathcal{R}$-modules
CWcomp	Category of CW complexes
Ext	Ext functor
Fil	Category of filtrations
$\mathbf{Fun}(\mathcal{C}, \mathcal{D})$	Category of functors between categories $\mathcal{C}$ and $\mathcal{D}$
$\mathbf{Hom}_{\mathcal{C}}(X, Y)$	Space of morphisms between objects $X, Y \in \mathcal{C}$
$\mathcal{D}(\mathcal{R})$	Derived category
$\mathrm{ob}(\mathcal{C})$	Objects of category $\mathcal{C}$
$\mathcal{R}$-**mod**	Category of $\mathcal{R}$-modules
Set	Category of sets
Top	Category of topological spaces
Tor	Tor functor
Vect	Category of vector spaces
∂	Boundary map
$\chi(X)$	Euler characteristic of X
$\oplus$	Direct sum
$\otimes_{\mathcal{R}}$	Tensor product with respect to a ring $\mathcal{R}$
$B^n(X_*)$	n-coboundaries of X_*

$B_n(X_*)$	n-boundaries of X_*
$C_*^{cell}(X)$	Cellular chain complex associated to X
$H^n(X_*)$	n^{th} cohomology of X_*
$H_n(X_*)$	n^{th} homology of X_*
$S^*(X_*)$	Cochain complex on X_*
$S_n(X_*)$	Free abelian group on X_*
$Z^n(X_*)$	n-cocycles of X_*
$Z_n(X_*)$	n- cycles of X_*
$\bar{A}$	Closure of the set A
$\mathbb{F}_p$	Finite field of order p
$\operatorname{Im} f$	Image of the map f
$\ker f$	Kernel of the map f
$\mathbb{D}^n$	n-dimensional disk
$\mathbb{S}^n$	n-dimensional sphere
$\mathbb{T}^n$	n-dimensional torus
$\mathbb{N}$	Non-negative integers
$\mathbb{R}^n$	n-dimensional real vector space
$\mathbb{RP}^n$	n-dimensional real projective space
$\mathbb{Z}^n$	Integer lattice
$\mathbb{Z}_+$	Positive integers
A°	Interior of the set A
$C(X,Y)$	Space of continuous functions from X to Y
$U \Subset X$	U is open subset of X

Chapter 1

Homology

1.1 Prologue

Question 1.1.1. How can one distinguish two topological spaces (up to homeomorphism or homotopy equivalence)?

To answer the Question (1.1.1), one often defines topological invariants, as quantities associated with topological spaces that are stable under homeomorphism (homotopy equivalence). The hope is that the invariants encode enough information to distinguish a fair number of spaces. We could associate 0 to every topological space to get an invariant but it is so trivial that one can't answer the Question 1.1.1.

In these notes, we will distinguish two topological spaces (and later smooth manifolds) by counting the number of holes, which turns out to be a topological invariant. For instance, a circle has one hole, the figure 8 has two holes, and the figure Y has no holes. The main objective of these notes is to give a precise definition of holes and an algorithm to compute them. The fundamental group of a space X is a way to count the number of holes but we won't cover them in these notes. Instead, we will study (co)homology groups.

As a teaser, we will give an algorithm to compute the *first homology group* $H_1(X)$ where X is $\mathbb{S}^1$, the figure Y, the boundary of a triangle, and a solid triangle even before formally defining $H_1(X)$. Then we will see how $H_1(X)$ recovers the information about the number of holes in X.

First, let's compute $H_1(\mathbb{S}^1)$.

- Decompose $\mathbb{S}^1$ into a directed graph as in Figure 1.1.
- Define an abelian group homomorphism $\partial : \mathbb{Z}\{f, g\} \to \mathbb{Z}\{x, y\}$ such that $f \mapsto y - x$ and $g \mapsto y - x$ where $\mathbb{Z}\{f, g\}$ and $\mathbb{Z}\{x, y\}$ are free abelian groups, i.e., $\mathbb{Z}\{f, g\}$ contains

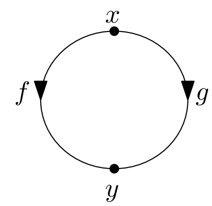


Figure 1.1: Directed circle

elements of the form $af + bg$ with $a, b \in \mathbb{Z}$. Intuitively, ∂ takes a directed edge and subtracts the *source* from the *target.*

- Finally, define $H_1(\mathbb{S}^1) := \ker(\partial) \cong \mathbb{Z}\{f - g\}$. Since $\mathbb{Z}$ has one generator, we say that the circle has one hole.

Second, let's compute $H_1(Y)$.

- Decompose the figure Y into a directed graph as in Figure 1.2

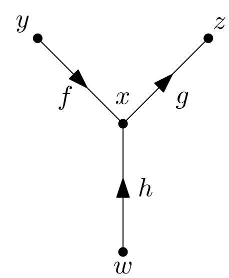


Figure 1.2: Figure Y

- Define $\partial : \mathbb{Z}\{f, g, h\} \to \mathbb{Z}\{x, y, z, w\}$ such that $f \mapsto x - y$, $g \mapsto z - x$ and $h \mapsto x - z$.
- As in the previous example, define $H_1(Y) := \ker \partial = \{0\}$. Since the trivial group has no generator, we conclude that Y has no hole.

Before proceeding further, let's pause to see how the algorithm works:

1. We decompose (*triangulate*) a geometric space X into a combinatorial one.
2. We associate some groups to the combinatorial object and do some algebra to get an algebraic object that seems to encode the information about the number of holes in X.

Third, consider a hollow triangle T as in Figure 1.3. Define a homomorphism $\partial : \mathbb{Z}\{f, g, h\} \to \mathbb{Z}\{a, b, c\}$ such that $f \mapsto b-a$, $g \mapsto c-b$ and $h \mapsto c-a$. Then $H_1(T) := \ker(\partial) = \mathbb{Z}\{f+g-h\}$, i.e., the hollow triangle has one hole.

Finally, consider a solid triangle S as in Figure 1.4. Define $\mathbb{Z}\{A\} \xrightarrow{\partial_2} \mathbb{Z}\{f, g, h\} \xrightarrow{\partial_1} \mathbb{Z}\{a, b, c\}$ such that $A \mapsto f+g-h$ and ∂_2 is same as ∂ in the case of the hollow triangle. Then $H_1(S) :=$

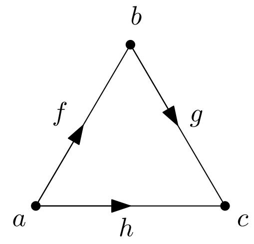


Figure 1.3: Hollow triangle

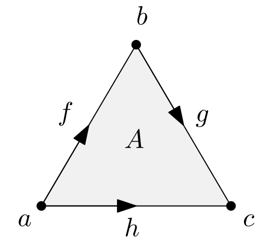


Figure 1.4: Solid triangle

$\ker(\partial_1)/\operatorname{Im}(\partial_2) = \{0\}$ where $\ker(\partial_1)$ denotes "cycles" and $\operatorname{Im}(\partial_2)$ denotes "boundaries." Therefore, the solid triangle has no holes.

Note that in all of the examples above we considered two-dimensional holes. On the other hand, $\mathbb{S}^2$ has a three-dimensional hole. In any case, we will devote the rest of these notes to making precise the definition n-dimensional holes. In this chapter, we will define homology.

1.2 Semisimplicial sets

In this section, we will define homology on spaces already equipped with a *triangulation*, before addressing the first point 1 in our algorithm to triangulate a geometric space into a combinatorial one.

Definition 1.2.1. An *n-simplex* $(e_0, \ldots, e_n)$ is the convex hull of $n+1$ points $e_0, \ldots, e_n \in \mathbb{R}^N$, i.e.,

$$(e_0, \ldots, e_n) \coloneqq \{\sum_i \lambda_i e_i : \lambda_0 \geq 0, \sum \lambda_i = 1\}.$$

If $\{e_i\}_{0\leq i\leq n} \in \mathbb{R}^{n+1}$ are the standard basis vectors then the n-simplex is known as the *standard n-simplex* and is denoted by Δ^n.

The following is an abstract way of defining a space equipped with a triangulation.

Definition 1.2.2. A *semisimplicial set* X_* is a sequence of sets $\{X_n\}_{n\in\mathbb{N}_0}$ where X_n is the set of *n-simplicies* equipped with functions $d_0, \ldots, d_n : X_n \to X_{n-1}$ for $n \geq 1$ satisfying the *simplicial identity:*

$$d_i \circ d_j = d_{j-1} \circ d_i \tag{1.2.1}$$

whenever $i < j$ and both sides of the equation make sense.

Example 1.2.3. 1. Suppose $X_n = \emptyset$ for $n \geq 2$. Then a semisimplicial set is a directed graph. In fact, X_0 is the set of vertices and X_1 is the set of edges, and the functions $d_0, d_1 : X_1 \to X_0$ assign to each edge its "target" and "source" respectively. For instance, the hollow triangle in Figure 1.3 is a semisimplicial set where $X_0 \coloneqq \{a, b, c\}$, $X_1 \coloneqq \{f, g, h\}$ and $X_n = \emptyset$ for $n \geq 2$.

2. The solid triangle in Figure 1.3 is a semisimplicial set where $X_0 \coloneqq \{a, b, c\}$, $X_1 \coloneqq \{f, g, h\}$ and $X_2 \coloneqq \{A\}$, while $X_n = \emptyset$ for $n \geq 3$. Moreover, $d_0(f) \coloneqq b$, $d_0(g) \coloneqq c$, $d_0(h) \coloneqq c$ and $d_1(f) \coloneqq a$, $d_1(g) \coloneqq b$, $d_1(h) \coloneqq a$. Using the simplicial identity, we get $d_0(A) = g$, $d_1(A) = h$ and $d_2(A) = f$. Intuitively, to get $d_0(A)$ we delete a from A to get the edge g. Similarly, to get $d_1(A)$, we delete b and get h. Here we have exploited the fact that a, b, c come with direction. Hopefully, the idea of deletion to get boundary maps will be clearer with more examples.

For a simplicial set X_*, we write $S_n(X_*)$ to mean the free abelian group on X_n. Further, we define the boundary map $\partial_n : S_n(X_*) \to S_{n-1}(X_*)$ such that for any $\sigma \in X_n$:

$$\partial_n(\sigma) \coloneqq \sum_{k=0}^{n} (-1)^k d_k(\sigma).$$

As a convention, we set $S_{-1} \coloneqq 0$ and $\partial_0 : S_0(X_*) \to 0$ is the zero homomorphism. Further, we will write ∂ instead of ∂_n when the context is clear.

Definition 1.2.4. Fix a semisimplicial set X_*. The *n-cycles* $Z_n(X_*)$ is the group $\ker(\partial_n)$. The *n-boundaries* $B_n(X_*)$ is the group $\mathrm{Im}(\partial_{n+1})$ and the *n^{th} homology group* $H_n(X_*)$ is the quotient $Z_n(X_*)/B_n(X_*)$.

Definition 1.2.5. Suppose X is a topological space.

1. Set $\mathbf{Sing}_n(X) \coloneqq C(\Delta^n; X)$, the space of continuous functions from Δ^n to X.[1]

   Note that there are maps $d_i : \mathbf{Sing}_n(X) \to \mathbf{Sing}_{n-1}(X)$ obtained by "forgetting" the vertex e_i in Δ^n. Check that $\mathbf{Sing}_*(X)$ with the maps d_i is a semisimplicial set.

2. Set $S_n(X) \coloneqq S_n(\mathbf{Sing}_* X)$.

3. The *n-cycles of* X is the group $Z_n(X) \coloneqq Z_n(\mathbf{Sing}_* X)$.

4. The *n-boundaries of* X is the group $B_n(X) \coloneqq B_n(\mathbf{Sing}_* X)$.

5. The *n^{th} homology* is the group $H_n(X) \coloneqq H_n(\mathbf{Sing}_* X)$.

To summarize, we have produced the following operations so far:

$$\mathbf{Top} \overset{\mathbf{Sing}_*}{\rightsquigarrow} \text{Semisimplicial sets} \overset{S_n,\ B_n,\ Z_n,\ H_n}{\rightsquigarrow} \mathbf{Ab} \tag{1.2.2}$$

where **Top** is the space of topological spaces and **Ab** is the space of abelian groups.

Remark 1.2.6. If $f : X \to Y$ is a continuous map of topological spaces and $\mathbf{Sing}_n(X) \ni \sigma : \Delta^n \to X$ then the composition $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} X$ is an element of $\mathbf{Sing}_n(Y)$. In fact, we have a map $\mathbf{Sing}_n(f) : \mathbf{Sing}_n(X) \to \mathbf{Sing}_n(Y)$ induced by the composition map.

[1] The term **Sing** stands for singular since we are allowing *singular domains*. In fact, the n-simplicies are singular at the vertices.

1.3 Category theory

In this section, we will formalize using categorical language the operations in (1.2.2) (cf. Theorem 1.3.16). For more details on the category theory see [Lei14], [Mac88] or [Rie17].

Definition 1.3.1. A *category* $\mathcal{C}$ consists of the following data:

- a *class*[2] $\mathrm{ob}(\mathcal{C})$ of *objects*,
- for every $X, Y \in \mathrm{ob}(\mathcal{C})$, a set of *morphisms* $\mathbf{Hom}_{\mathcal{C}}(X, Y)$,
- for every $X \in \mathrm{ob}(\mathcal{C})$, an identity morphism $\mathbb{1}_X \in \mathbf{Hom}_{\mathcal{C}}(X, X)$,
- for every X, Y and $Z \in \mathrm{ob}(\mathcal{C})$, a composition operation $\mathbf{Hom}_{\mathcal{C}}(X, Y) \times \mathbf{Hom}_{\mathcal{C}}(Y, Z) \to \mathbf{Hom}_{\mathcal{C}}(X, Z)$ such that $(f, g) \mapsto g \circ f$. Further, the composition must satisfy $h \circ (g \circ f) = (h \circ g) \circ f$. Here, $\mathbb{1}_Y \circ f = f$ and $f \circ \mathbb{1}_X = f$ for every $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$.

Example 1.3.2.

1. **Set** is the category where $\mathrm{ob}(\mathbf{Set})$ consists of all sets, and if $X, Y \in \mathrm{ob}(\mathbf{Set})$ then $\mathbf{Hom}_{\mathbf{Set}}(X, Y) \coloneqq \{\text{functions from } X \to Y\}$
2. **Ab** is the category where $\mathrm{ob}(\mathbf{Ab})$ consists of all abelian groups and $\mathbf{Hom}_{\mathbf{Ab}}(X, Y)$ consists of all group homomorphisms from X to Y.
3. **Top** is the category where $\mathrm{ob}(\mathbf{Top})$ consists of all topological spaces and $\mathbf{Hom}_{\mathbf{Top}}(X, Y) \coloneqq C(X; Y)$.
4. $\mathbf{Vect}_{\mathbb{R}}$ is the category of real vector spaces where morphisms are linear transformations. $\mathbf{Vect}_{\mathbb{R}}$ is also called the category of modules over real numbers.

For any category $\mathcal{C}$, we often write $X \in \mathcal{C}$ as a shorthand for $X \in \mathrm{ob}(\mathcal{C})$ and $f : X \to Y$ for $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$.

Definition 1.3.3. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$ and $g \circ f = \mathbb{1}_X$.

Example 1.3.4.

1. An isomorphism in **Set** is a bijection.
2. An isomorphism in **Top** is a homeomorphism.

Proposition 1.3.5. *Suppose $f : X \to Y$ is an isomorphism. Then the inverse $g : Y \to X$ is unique.*

Proof. Suppose that $g' : Y \to X$ were another inverse. Then

$$(g \circ f) \circ g' = \mathbb{1}_x \circ g' = g' = g \circ (f \circ g') = g \circ \mathbb{1}_y = g.$$

□

[2] We have used the word class (informally) to avoid the set theoretical paradoxes like the existence of a set of all sets. However, one can talk about a class of all sets. The idea of class can be formalized using Grothendieck universes. For our purpose, we don't have to worry about the foundation.

Definition 1.3.6. A *functor* $F : \mathcal{C} \to \mathcal{D}$ *between two categories* $\mathcal{C}$ *and* $\mathcal{D}$ is a map $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ such that for all $X, Y \in \mathcal{C}$ there is an induced map $F : \mathbf{Hom}_{\mathcal{C}}(X, Y) \to \mathbf{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Example 1.3.7.
1. For each $n \in \mathbb{N}_0$, there is a functor $\mathbf{Sing}_n : \mathbf{Top} \to \mathbf{Set}$ that maps a topological space X to $C(\Delta^n; X))$ and $C(X; Y)$ to $C(\Delta^n; Y)$ via composition. Similarly, there is a functor $S_n : \mathbf{Top} \to \mathbf{Ab}$ that maps a topological space X to an abelian group $S_n(X)$. Moreover, there is a functor $H_n : \mathbf{Top} \to \mathbf{Ab}$ that maps a topological space X to an abelian group $H_n(X)$.
2. There is a (huge) category $\mathbf{Cat}$ where $\mathrm{ob}(\mathbf{Cat})$ consists of all categories and morphisms between categories are functors $f : \mathcal{C} \to \mathcal{D}$.
3. We can view the functor $S_n : \mathbf{Top} \to \mathbf{Ab}$ as a composition of $\mathbf{Sing}_n : \mathbf{Top} \to \mathbf{Set}$ with $\mathrm{Free} : \mathbf{Set} \to \mathbf{Ab}$ where Free maps a set X to the free abelian group generated by X.

Definition 1.3.8. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\Theta : F \to G$ *between the two functors* consists of maps $\Theta_X : F(X) \to G(X)$ for all $X \in \mathcal{C}$ such that for all $f \in \mathbf{Hom}_{\mathcal{C}}(X, Y)$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\Theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Theta_Y"] & G(Y). \end{tikzcd}$$

Further, Θ is a *natural isomorphism* if Θ_X is an isomorphism for all $X \in \mathcal{C}$.

Example 1.3.9. If $n \in \mathbb{N}$ and $0 \leq i \leq n$ then there is a natural transformation $d_i : \mathbf{Sing}_n \to \mathbf{Sing}_{n-1}$ between the functors $\mathbf{Sing}_n, \mathbf{Sing}_{n-1} : \mathbf{Top} \to \mathbf{Set}$. In fact, check that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Sing}_n(X) \arrow[r, "d_i"] \arrow[d, "\mathbf{Sing}_n(f)"] & \mathbf{Sing}_{n-1} X \arrow[d, "\mathbf{Sing}_n(f)"] \\ \mathbf{Sing}_n(Y) \arrow[r, "d_i"] & \mathbf{Sing}_{n-1}(Y). \end{tikzcd}$$

Definition 1.3.10. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories such that $\mathcal{C}$ has a set of objects. Then the *functor category* $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ has the following data:

- $\mathrm{ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})) = \mathbf{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ which consists of functors from $\mathcal{C}$ to $\mathcal{D}$, and
- morphsims are natural transformations.

Definition 1.3.11. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ consists of

- $\mathrm{ob}(\mathcal{C}^{op}) = \mathrm{ob}(\mathcal{C})$ and

- $\mathbf{Hom}_{\mathcal{C}^{op}}(C_1, C_2) = \mathbf{Hom}_{\mathcal{C}}(C_2, C_1)$.

If $f \in \mathbf{Hom}_{\mathcal{C}}(C_2, C_1)$, we use f^{op} to denote the corresponding element of $\mathbf{Hom}_{\mathcal{C}^{op}}(C_1, C_2)$. The composition $(f \circ g)^{op}$ equals to $g^{op} \circ f^{op}$.

Example 1.3.12. Note that every vector space V has a dual vector space $V^* := \underline{\mathbf{Hom}}_{\mathbf{Vect}_{\mathbb{R}}}(V, \mathbb{R})$, the vector space of linear functionals on V. If $f \in \mathbf{Hom}_{\mathbf{Vect}_{\mathbb{R}}}(W, V)$ then for every $g \in V^*$ we get an element $g \circ f \in W^*$. Therefore, a linear map $f : W \to V$ induces a map $V^* \to W^*$. Precisely, there is functor $()^* : \mathbf{Vect}_{\mathbb{R}}^{op} \to \mathbf{Vect}_{\mathbb{R}}$.

For any $n \in \mathbb{N}_0$, let $[n]$ be the totally ordered set $\{0, \ldots, n\}$. We write Δ_{inj} to denote the category with $\mathrm{ob}(\Delta_{inj}) := \{[n]\}_{n \in \mathbb{N}}$. Further, $\mathbf{Hom}_{\Delta_{inj}}([a], [b])$ is the set of all order-preserving injective functions $f : \{0, \ldots, a\} \to \{0, \ldots, b\}$, i.e., $f(x) < f(y)$ whenever $x < y$. For instance, there are three maps in Δ_{inj} from [1] to [2].

Definition 1.3.13 (Proposition)**.** A *semisimplicial set* is a functor $\Delta_{inj}^{op} \to \mathbf{Set}$.

Remark 1.3.14.
1. The Definitions 1.2.2 and 1.3.13 are equivalent. In fact, given a semisimplicial set $\{X_n\}_{n \in \mathbb{N}}$ with maps $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$, we define the corresponding functor $F : \Delta_{inj}^{op} \to \mathbf{Set}$ as $F([n]) := X_n$ and the element in $\mathbf{Hom}_{\Delta_{inj}^{op}}([n], [n-1])$ that forgets $i \in [n]$ corresponds to $d_i : X_n \to X_{n-1}$. Check that the simplicial identities (1.2.1) are compatible with the functoriality of F.
2. When $X_* = S_*(\mathbf{Sing}_*(X))$ for a topological space X, then the map that forgets $i \in [n]$ corresponds to the map $\mathbf{Sing}_n(X) \ni \sigma \mapsto d_i(\sigma)$ where $d_i\sigma$ is the composition $\Delta^{n-1} \to \Delta^n \to X$ where $\Delta^{n-1} \to \Delta^n$ is an affine map that sends each vertex in Δ^{n-1} to a vertex in Δ^n forgetting e_i.

Definition 1.3.15. The *category of semisimplicial sets* is the functor category $\mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set})$.

The following theorem allows us to view the quantities in Definition 1.2.5 using the categorical language:

Theorem 1.3.16. *There are functors*

$$\mathbf{Sing} : \mathbf{Top} \to \mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set}), \tag{1.3.1a}$$

$$S_n, Z_n, B_n, H_n : \mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set}) \to \mathbf{Ab}. \tag{1.3.1b}$$

1.4 Chain complexes

In Theorem (1.3.16), instead of considering the functor S_n (and similarly Z_n, B_n, H_n), we can consider the functor S_* that takes a semisimplicial set X_* and outputs a sequence of abelian groups $\{S_n(X_*)\}$ and associated boundary maps. In this section, we will formalize the functor S_* (cf. Theorem 1.4.6) and define homology for a sequence of abelian groups (*chain complex*).

Definition 1.4.1. Let **Fil**[3] denote the category such that

- for every $n \in \mathbb{Z}$ there is one object X_n,
- if $m < n$ then there is no morphism $X_m \to X_n$, and
- if $m \geq n$ then there is a unique morphism $X_m \to X_n$.

We can represent objects of **Fil** by $\bullet$ and morphisms by arrows. Then **Fil** can be visualized as in Figure 1.5. The loops at each point correspond to the identity map, and the arrows compose.

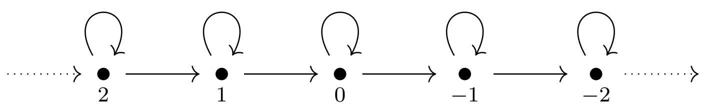


Figure 1.5: **Fil**

Remark 1.4.2. A functor $A : \mathbf{Fil} \to \mathbf{Ab}$ is a sequence of abelian groups $\{A_n\}_{n\in\mathbb{Z}}$ with some maps $\partial_n : A_n \to A_{n-1}$.

Definition 1.4.3. A *chain complex of abelian groups* is a functor $A : \mathbf{Fil} \to \mathbf{Ab}$ with the property that $\partial_i \circ \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$.

Example 1.4.4. Let X_* be a semisimplicial set. Then there is a chain complex $\dashrightarrow S_1(X_*) \xrightarrow{\partial_1} S_0(X_*) \xrightarrow{\partial_0} 0 \to 0 \dashrightarrow$ where $\partial_i(\sigma) = \sum_k (-1)^k d_k(\sigma)$.

Definition 1.4.5. The *category* **chAb** *of chain complexes* is the functor category $\mathbf{Fun}(\mathbf{Fil}, \mathbf{Ab})$.

Explicitly, **chAb** consists of chain complexes (functors $\mathbf{Fil} \to \mathbf{Ab}$) and morphisms are natural transformations between chain complexes. Here, a natural transformation between two functors $A, B : \mathbf{Fil} \to \mathbf{Ab}$ consists of maps $A_i \to B_i$ such that each square in the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & A_{-1} \arrow[r, "\partial_0"] \arrow[d] & A_0 \arrow[r, "\partial_1"] \arrow[d] & A_1 \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & B_{-1} \arrow[r, "\partial_0"] & B_0 \arrow[r, "\partial_1"] & B_1 \arrow[r, dotted] & \cdots
\end{tikzcd}.$$

Theorem 1.4.6. *There is a functor $S_* : \mathbf{Fun}(\Delta^{op}_{inj}, \mathbf{Set}) \to \mathbf{chAb}$. Moreover, there are functors $Z_n, B_n, H_n : \mathbf{chAb} \to \mathbf{Ab}$ for all $n \in \mathbb{Z}$.*

Using Theorem 1.4.6, we can define the homology functor $H_n : \mathbf{Top} \to \mathbf{Ab}$ as the composite of three functors $\mathbf{Sing}_* : \mathbf{Top} \to \mathbf{Fun}(\Delta^{op}_{inj}, \mathbf{Set})$ and $S_* : \mathbf{Fun}(\Delta^{op}_{inj}, \mathbf{Set}) \to \mathbf{chAb}$ and $H_n : \mathbf{chAb} \to \mathbf{Ab}$.

[3]The term **Fil** stands for filtration.

Chapter 2

Computing homology

Recall that our goal is to compute $H_i(X)$ for various $X \in \mathbf{Top}$ and $i \in \mathbb{Z}$ as well as to prove its topological invariance. To simplify the program, we view H_n as a functor viewed as the composite of $\mathbf{Sing} : \mathbf{Top} \to \mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set})$, $S_* : \mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set}) \to \mathbf{chAb}$ and $H_i : \mathbf{chAb} \to \mathbf{Ab}$. Then we want to understand the behavior of H_i based on operations in **Top** so that we can compute homologies of topological spaces formed via these operations. Precisely, we want to answer the following questions:

Question 2.0.1. 1. What are the properties of $H_i(f_1), H_i(f_2) : H_i(X) \to H_i(Y)$ for given maps $f_j : X \to Y$ in **Top**?

2. How does H_i behave under the following topological operations:
   - inclusion,
   - product,
   - quotient,
   - union, and
   - wedge sum?

Note that in the Question 2.0.1 1, the functoriality of H_* implies that $H_i(f_j)$ is an isomorphism whenver f_j is a homeomorphism. We will see in §2.2 that $H_i(f_j)$ are the same map whenever f_j are homotopic.

As a first answer to the Question 2.0.1 2, note that an inclusion of topological spaces induces inclusion of homological groups. Then in §2.3, §2.4, and §2.5 we will study how H_i behaves under quotient operation. Further, in §2.6 we will study the homology of the union of spaces. One can use the Excision Principle (cf. Theorem 2.4.1) to compute the homology of the wedge sum of spaces. However, we will postpone the case of products to subsequent chapters (cf. Theorem 4.6.1).

2.1 Homology of star-shaped spaces

Before answering the Question 2.0.1, we will warm up by computing the homologies of *star-shaped spaces.*

Definition 2.1.1. A set $X \subset \mathbb{R}^n$ is *star-shaped with respect to a point* $b \in X$ if for every point $x \in X$, the interval $\{tb + (1-t)x \mid t \in [0,1]\}$ is a subset of X.

Example 2.1.2. Any convex region is star-shaped with respect to any of its points. Note that any convex region has no hole. In fact, any star-shaped region has no hole as proven in the following theorem.

Theorem 2.1.3. *If X is star-shaped with respect to a point $b \in X$, then*

$$H_i(X) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{2.1.1}$$

Remark 2.1.4. Check that H_0 counts the number of path components of the space. Further, H_i "counts" the number of higher dimensional holes.

As a simple case of Theorem 2.1.3, consider X to be a topological space with one point p. For each n, $C(\Delta^n, X)$ is a one element set say $\{a_n : \Delta^n \to X\}$, i.e., $\mathbf{Sing}_n(X) = \{a_n\}$. Note that $d_i : \mathbf{Sing}_n(X) \to \mathbf{Sing}_{n-1}(X)$ maps a_n to a_{n-1}. Then $S_*(X)$ is

$$\dashrightarrow \mathbb{Z}\{a_1\} \xrightarrow{\partial_1} \mathbb{Z}\{a_0\} \to 0 \to 0 \dashrightarrow,$$

where $\partial_n(a_n) = 0$ for odd n and $\partial_n(a_n) = a_{n-1}$ for even n. In particular, $S_*(X)$ is isomorphic in **chAb** to $\dashrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to 0 \dashrightarrow$. Therefore, $H_0(X) = \ker(\partial_0)/\operatorname{Im}(\partial_1) = \mathbb{Z}/0 \cong \mathbb{Z}$. Further, $H_1(X) = \ker(\partial_1)/\operatorname{Im}(\partial_2) = \mathbb{Z}/\mathbb{Z} \cong 0$. And $H_2(X) = \ker(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 \cong 0$. In fact,

$$H_i(\{p\}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Before proving Theorem 2.1.3, let's define a notion of *(chain) homotopy* between two chain complexes. In §2.2, we will see a topological context where we run across chain homotopy. Anyway, Proposition 2.1.6 says that homology is invariant under chain homotopy. Then to prove Theorem 2.1.3, it suffices to construct a chain homotopy between chain complexes associated with a point and a star-shaped region.

Definition 2.1.5. Let C_* and D_* be chain complexes and $f_0, f_1 : C_* \to D_*$ be maps of chain complexes. A *chain homotopy* $h : f_0 \simeq f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$ such that $\partial h + h\partial = f_1 - f_0$, see the diagram below.[1] We say that f_0 *and* f_1 *are chain homotopic* if there exists an $h : f_0 \simeq f_1$.

[1]The diagram is not a commutative diagram.

$$\begin{tikzcd}
\arrow[r, dotted, "\partial"] & C_1 \arrow[r, "\partial"] \arrow[d, "f_0\ f_1"] & C_0 \arrow[r, "\partial"] \arrow[d, "f_0\ f_1"] \arrow[dl, dotted, "h"] & C_{-1} \arrow[r, dotted, "\partial"] \arrow[d, "f_0\ f_1"] \arrow[dl, dotted, "h"] & {} \\
\arrow[r, dotted, "\partial"] & D_1 \arrow[r, "\partial"] & D_0 \arrow[r, "\partial"] & D_{-1} \arrow[r, dotted, "\partial"] & {} .
\end{tikzcd}$$

Proposition 2.1.6. *Suppose $f_0, f_1 : C_* \to D_*$ are chain homotopic (say via $h : f_0 \simeq f_1$). Then $H_n(f_0)$ and $H_n(f_1)$ are identical group homomorphisms $H_n(C_*) \to H_n(D_*)$.*

Proof. Suppose $c \in Z_n(C_*)$ is an n-cycle, i.e., $\partial c = 0$. We must show that $f_1(c) - f_0(c) \in Z_n(D_*)$ is a boundary. In fact, $f_1(c) - f_0(c) = (\partial h + h\partial)c = \partial hc$. □

Proof of Theorem 2.1.3. Let p be a topological space with one point. Recall that $S_*(p)$ is isomorphic to the chain complex where $X_0 = \mathbb{Z}$ and $X_i = 0$ else. Note that the trivial maps $X \to p$ and $p \mapsto b \in X$, where b is the point with respect to which X is star-shaped, induce maps of chain complexes $\epsilon : S_*(X) \to S_*(p)$ and $\eta : S_*(p) \to S_*(X)$ such that there are two commutative diagrams

$$\begin{tikzcd}
\arrow[r, dotted, "\partial"] & S_1(X) \arrow[r, "\partial"] \arrow[d, "\epsilon_1"] & S_0(X) \arrow[r, "\partial"] \arrow[d, "\epsilon_0"] & 0 \arrow[r, dotted, "\partial"] \arrow[d, "\epsilon_{-1}"] & {} \\
\arrow[r, dotted, "\partial"] & 0 \arrow[r, "\partial"] & \mathbb{Z} \arrow[r] & 0 \arrow[r, dotted, "\partial"] & {}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\arrow[r, dotted, "\partial"] & 0 \arrow[r, "\partial"] \arrow[d, "\eta_1"] & \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "\eta_0"] & 0 \arrow[r, dotted, "\partial"] \arrow[d, "\eta_{-1}"] & {} \\
\arrow[r, dotted, "\partial"] & S_1(X) \arrow[r, "\partial"] & S_0(X) \arrow[r, "\partial"] & 0 \arrow[r, dotted, "\partial"] & {} ,
\end{tikzcd}$$

where, for $p_i \in X$ and $a_i \in \mathbb{Z}$, we set $\epsilon_0(\sum a_i p_i) \coloneqq \sum a_i$. Further, $\eta_0(1) = b$.

Since $S_*(p)$ has the homology data as in (2.1.1), it suffices to prove that $H_i(\epsilon) : H_i(S_*(X)) \to H_i(S_*(p))$ and $H_i(\eta) : H_i(S_*(p)) \to H_i(S_*(X))$ are inverse maps of abelian groups. Note that $\epsilon \circ \eta : S_*(p) \to S_*(p)$ is identity. Therefore, for all i, we have $H_i(\epsilon) \circ H_i(\eta) = H_i(\epsilon \circ \eta) = \mathbb{1}_{H_i(S_*(p))}$.

Conversely, define a chain homotopy $h : \eta \circ \epsilon \simeq \mathbb{1}_{S_*(X)}$, i.e., maps $h_q : S_q(X) \to S_{q+1}(X)$ such that, for any $\sigma \in S_q(X)$, the map $h_q(\sigma) \in S_{q+1}(X)$ is defined as follows

$$\Delta^{q+1} \ni (t_0, \dots, t_{q+1}) \mapsto t_0 b + (1 - t_0)\sigma\left(\frac{(t_1, \dots, t_{q+1})}{1 - t_0}\right). \tag{2.1.2}$$

When $t_0 = 1$, the image is understood to be b. See Figure 2.1 for a geometrical insight. In any case, Proposition 2.1.6 implies that $H_i(\eta) \circ H_i(\epsilon) = H_i(\eta \circ \epsilon) = H_i(\mathbb{1}_{S_*(X)}) = \mathbb{1}_{H_i(S_*(X))}$ which proves (2.1.1) if we can show that h is in fact a chain homotopy.

Assume that $q > 0$. For any $\sigma \in S_q(X)$, we have $d_0(h_q(\sigma)) = \sigma$ and $d_i(h_q(\sigma)) = h_{q-1}(d_{i-1}\sigma)$ for $i \geq 1$. Therefore,

$$\partial h_q \sigma = \sum_{k=0}^{q+1} (-1)^k d_k(h_q \sigma) = \sigma - \sum_{k=0}^{q} (-1)^k h_{q-1}(d_k \sigma) = \sigma - h_{q-1}\partial\sigma.$$

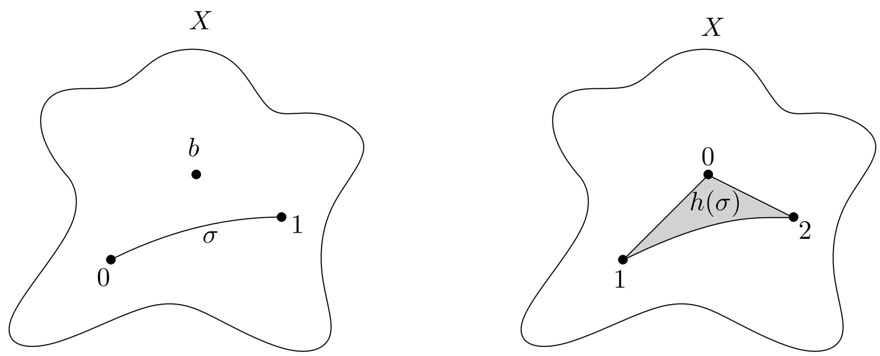


Figure 2.1: $h_q(\sigma)$

On the other hand, for $q = 0$, $h_{-1}\partial p_i = 0$ for any $p_i \in X$. Further, $h_0 p_i(t_0) = t_0 b + (1 - t_0)p$. Therefore, $\partial h_0 p_i = p_i - b = p_i - \eta(\epsilon(p_i))$. In any case,

$$\partial h + h\partial = \mathbb{1}_{S_*(X)} - \eta \circ \epsilon$$

which proves that h is a chain homotopy. $\square$

2.2 Topological invariance of homology

In Definition 2.1.5, we introduced the concept of chain homotopy $h : f_0 \simeq f_1$ between the maps $f_0, f_1 : C_* \to D_*$ in **chAb** and in Proposition 2.1.6 we proved that chain homotopic maps induce the same map in homology. Explicitly, if there exists $h_1 : f_0 \simeq f_1$ then $H_n(f_0) = H_n(f_1) : H_n(C_*) \to H_n(D_*)$. In this section, we will introduce a topological situation, *homotopy equivalence*, that gives rise to chain homotopy, partially answering the following question and therefore answering the Question 2.0.1 1:

Question 2.2.1. If $f, g : X \to Y$ maps in **Top** when is $S_*(f)$ chain homotopic to $S_*(g)$?

In the case when $S_*(f)$ and $S_*(g)$ are chain homotopic, we know that $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$.

Definition 2.2.2. A *homotopy h between maps $f, g : X \to Y$ in* **Top** is a continuous map $h : X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. If such h exists, we say that f and g are *homotopic* and write $f \simeq g$.

Remark 2.2.3. Geometrically, a homotopy is a continuous deformation of f to g. In fact, the second coordinate in the domain of h can be thought of as time.

Theorem 2.2.4. *Suppose $f, g : X \to Y$ are homotopic maps. Then $S_*(f)$ and $S_*(g)$ are chain homotopic, i.e., $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.*

Proof. Exercise or see Section 6 of [Mil16]. $\square$

Remark 2.2.5. Let $f_1, f_2, f_3 : X \to Y$ be maps in **Top**. Suppose $h_{12} : X \times [0,1] \to Y$ is a homotopy from f_1 to f_2 and $h_{23} : X \times [0,1] \to Y$ from f_1 to f_2. Then we can define a homotopy from f_1 to f_3 as

$$h_{13}(x,t) \coloneqq \begin{cases} h_{12}(x,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ h_{23}(x,2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

In particular, the relation $f \simeq g$ of homotopy is an equivalence relation on $\mathbf{Hom}_{\mathbf{Top}}(X,Y)$.

Definition 2.2.6. The *homotopy category* $\mathbf{Ho}(\mathbf{Top})$ is a category with $\mathrm{ob}(\mathbf{Ho}(\mathbf{Top})) \coloneqq \mathrm{ob}(\mathbf{Top})$ and $\mathbf{Hom}_{\mathbf{Ho}(\mathbf{Top})}(X,Y) \coloneqq \mathbf{Hom}_{\mathbf{Top}}(X,Y)/\simeq$.

Remark 2.2.7. There is a canonical functor $\mathbf{Top} \to \mathbf{Ho}(\mathbf{Top})$ that sends each topological space to itself and maps between topological spaces to the equivalence class of homotopic maps.

Theorem 2.2.4 can be restated as follows:

Theorem 2.2.8. *There is a commutative diagram in* **Cat**

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "H_n"] \arrow[d] & \mathbf{Ab} \\ \mathbf{Ho}(\mathbf{Top}). \arrow[ur, "H_n"'] & \end{tikzcd}$$

Remark 2.2.9. The theorem is false for the functors $\mathbf{Sing}, S_*, Z_n, B_n$ since none of these functors factor through $\mathbf{Ho}(\mathbf{Top})$).

Definition 2.2.10. A continuous map $f : X \to Y$ in **Top** is a *homotopy equivalence* if it is an isomorphim in $\mathbf{Ho}(\mathbf{Top})$.

Concretely, f is a homotopy equivalence if there exists an "inverse" $g : Y \to X$ such that $g \circ f \simeq \mathbb{1}_X$ and $f \circ g \simeq \mathbb{1}_Y$. Note that an inverse g is unique up to homotopy.

Definition 2.2.11. We say that X and $Y \in \mathbf{Top}$ are *homotopy equivalent* if they are isomorphic in $\mathbf{Ho}(\mathbf{Top})$, i.e., there is a homotopy equivalence between X and Y.

Remark 2.2.12. Theorem 2.2.8 says that homology fails to distinguish homotopy equivalent spaces and homotopic maps.

Definition 2.2.13. An inclusion $A \hookrightarrow X$ in **Top** is a *deformation retract* if there exists a map $h : X \times [0,1] \to X$ in **Top** such that

1. $h(\bullet, 0) = \mathbb{1}_X$,

2. $h(\bullet, 1) \in A$, and

3. $h(\bullet, t)\big|_A = \mathbb{1}_A$ for all $t \in [0,1]$.

Remark 2.2.14. Check that if $A \hookrightarrow X$ is a deformation retract then A is homotopy equivalent to X.

Example 2.2.15. There are deformation retracts

- $\{0\} \hookrightarrow [0,1]$ or $[0,1)$,
- $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$, and
- $\{b\} \hookrightarrow X$ if X is star-shaped with respect to b.

For more examples of deformation retracts, see [Hat05].

Remark 2.2.16. Theorem 2.2.8 means that homology can't differentiate the "dimension" of homotopy equivalent spaces. For instance, a circle and a disc with a hole have the same homology but the disc with a hole locally looks like two-dimensional space.

2.3 Relative homology

Note that a deformation retraction $A \hookrightarrow X$ simplifies the computation of the homology of X to that of A. For general subspaces, we would like to answer the following question, answering which will give us a library of spaces whose homology can be easily computed:

Question 2.3.1. If $A \subset X$ is not a deformation retract, what is the relationship between $H_i(A), H_i(X)$ and $H_i(X/A)$?

Definition 2.3.2. We say that $C_* \subset D_*$ is an *inclusion of chain complexes* if there are inclusions $C_n \hookrightarrow D_n$ for all n. Define D_*/C_* to be the unique object of **chAb** such that the n^{th} group is D_n/C_n and the differentials ∂ are defined so that the following diagram commutes for all n:

$$\begin{tikzcd}
C_n \arrow[r, "\partial"] \arrow[d, hook] & C_{n-1} \arrow[d, hook] \\
D_n \arrow[r, "\partial"] \arrow[d] & D_{n-1} \arrow[d] \\
D_n/C_n \arrow[r, "\partial"] & D_{n-1}/C_{n-1}.
\end{tikzcd}$$

Note that the inclusion map $A \hookrightarrow X$ induces an inclusion $S_*(A) \hookrightarrow S_*(X)$.

Definition 2.3.3. Suppose $A \subset X$ is a subspace. The *relative homology* $H_n(X, A)$ is the n^{th} homology group of $S_*(X)/S_*(A)$.

In the rest of this section, we will work towards relating $H_n(X, A), H(X), H_n(A)$. First, let's restate the definition of relative homology in categorical language.

Notice that there is a category $\mathbf{Top_2}$ with

- $\mathrm{ob}(\mathbf{Top_2}) \coloneqq \{(X, A) \mid A \subset X \in \mathbf{Top}\}$
- $\mathbf{Hom}_{\mathbf{Top_2}}((X, A), (Y, B)) \coloneqq \{f \in C(X, Y) \mid f(A) \subset B\}$.

Then relative homology is a functor $H_n : \mathbf{Top_2} \to \mathbf{Ab}$.

Define a functor $F : \mathbf{Top} \to \mathbf{Top_2}$ such that $X \mapsto (X, \emptyset)$. Then there is a commutative diagram in $\mathbf{Cat}$ as follows:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "H_n"] \arrow[d, "F"'] & \mathbf{Ab} \\ \mathbf{Top_2} \arrow[ur, "H_n"'] & \end{tikzcd}$$

Definition 2.3.4. Two maps $f, g : (X, A) \to (Y, B)$ in $\mathbf{Top_2}$ are *homotopic* if there exists $h : X \times [0, 1] \to Y$ such that

- $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$
- $h(a, t) \subset B$ for all $a \in A$ and all $t \in [0, 1]$.

Remark 2.3.5. The diagram above says that a homotopy in $\mathbf{Top_2}$ induces an isomorphism in homology in $\mathbf{Top_2}$.

Before we can relate $H_n(X, A), H_n(X)$ and $H_n(A)$, we need some algebraic terminologies defined below.

Definition 2.3.6. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups with homomorphisms f and g is *exact (at B)* if $\ker(g) = \mathrm{Im}(f)$.

Example 2.3.7.

1. A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective.
2. A sequence $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective.

Definition 2.3.8. A longer sequence in $\mathbf{Ab}$ is *exact* if every three-term subsequence of adjacent objects is exact.

Example 2.3.9. A sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0$ is exact.

Definition 2.3.10. An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a *short exact sequence.*

Remark 2.3.11.

1. $S_*(\emptyset)$ is exact.

2. If X is a space with any non-zero homology group, then $S_*(X)$ is not exact. In fact, the homology groups measure the extent to which $S_*(X)$ fails to be exact.

Theorem 2.3.12 (Five Lemma). *Suppose there is a commutative diagram in* **Ab**

$$\begin{tikzcd} A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_1 \arrow[r, "d"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\ B_4 \arrow[r, "d"] & B_3 \arrow[r, "d"] & B_2 \arrow[r, "d"] & B_1 \arrow[r, "d"] & B_0 \end{tikzcd}$$

such that both rows are exact, and f_0, f_1, f_3, and f_4 are isomorphisms. Then f_2 is an isomorphism.

Proof.

- **Surjectivity:** We write $a_i(b_i)$ to mean generic elements in $A_i(B_i)$. Suppose $b_2 \in B_2$. Note that $db_2 = f_1 a_1$ for a unique a_1. Further, $f_0 da_1 = df_1 a_1 = ddb_2 = 0$ implies that $da_1 = 0$. Therefore, there exists a_2 such that $da_2 = a_1$. Note that $d(f_2 a_2 - b_2) = 0$, so there exists b_3 such that $db_3 = f_2 a_2 - b_2$. Since f_3 is an isomorphism, there exists a_3 such that $f_3 a_3 = b_3$. Now observe that $f_2(a_2 + da_3) = b_2$.
- **Injectivity:** Exercise.

□

Definition 2.3.13. A *short exact sequence* $0 \to A_* \to B_* \to C_* \to 0$ *of chain complexes* is a collection of short exact sequences $0 \to A_n \to B_n \to C_n \to 0$ satisfying the following commutative diagram for all n :

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\ B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\ C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[d] \\ 0 & 0 & 0. \end{tikzcd}$$

Example 2.3.14. If $A \subset X$ in **Top** then $0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$ is a short exact sequence.

Theorem 2.3.15 (Snake Lemma). *Suppose* $0 \to A_* \to B_* \to C_* \to 0$ *is a short exact sequence in* **chAb**. *Then there exists a long exact sequence*

$$
\begin{tikzcd}
\cdots \arrow[r, dotted] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, "\partial"] \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r, dotted] & \cdots .
\end{tikzcd}
$$

Proof. Exercise. □

Corollary 2.3.16. *Suppose $A \subset X$ in* **Top**. *Then there is a long exact sequence*

$$
\begin{tikzcd}
\cdots \arrow[r, dotted] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"] \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X,A) \arrow[r, dotted] & \cdots .
\end{tikzcd}
$$

Remark 2.3.17. Oftentimes, when computing homology we just need the mere existence of ∂ that connects the n^{th} homology to the $(n-1)^{th}$ homology rather than its explicit properties. For a geometric interpretation of ∂ in a special setting, see Remark 2.4.2.

Corollary 2.3.18 (Relative homology with respect to a point). *Consider a point $b \in X$. Then*

$$
H_n(X,\{b\}) \cong \begin{cases} H_n(X) & \text{if } n > 0, \\ H_0(X)/\mathbb{Z} & \text{if } n = 0. \end{cases} \tag{2.3.1}
$$

Proof. Consider the inclusion of a point $\{b\} \hookrightarrow X$. Then there is an exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r, dotted] & H_n(\{b\}) \arrow[r] & H_n(X) \arrow[r] & H_n(X,\{b\}) \arrow[dll, "\partial"] \\
& H_{n-1}(\{b\}) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X,\{b\}) \arrow[r, dotted] & \cdots .
\end{tikzcd}
$$

Note that when $n > 1$, $H_n(\{b\}) = 0$. Therefore, we get exact sequences $0 \to H_n(X) \to H_n(X,\{b\}) \to 0$ whence $H_n(X) \cong H_n(X,\{b\})$.

On the other hand, when $n = 1$ we have $0 \to H_1X \xrightarrow{i} H_1(X,\{b\}) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{j} H_0(X) = \mathbb{Z}\{\pi_0 X\}$ where $\pi_0 X$ is the connected components of X. Note that $j(1)$ is the path component of b, so the kernel of j is trivial. Therefore, the image of $H_1(X,\{b\})$ is 0 which implies that i is surjective. Note that i is also injective. Therefore, $H_1(X) \cong H_1(X,\{b\})$.

Finally, when $n = 0$ we have $H_0(\{b\}) \to H_0(X) \to H_0(X,\{b\}) \to 0$. Using exactness again, we get $H_0(X,\{b\}) = H_0X/H_0(\{b\})$. □

2.4 Excision

Note that Corollary 2.3.16 relates $H_n(X), H_n(A)$ and $H_n(X,A)$. In fact, when A is just a point $H_n(X,\{b\})$ can be recovered from $H_n(X)$. However, the relative homology group for general subspaces in its current form is technical to compute and provides no topological insight which makes it difficult to answer the Question 2.3.1. The main goal of the next couple of sections is to prove the following theorem that relates $H_n(X,A)$ to $H_n(X/A)$. Theorem 2.4.1 allows us to compute the homology of X if we know that of A and X/A.

Theorem 2.4.1 (Excision). *Consider a subspace $A \subset X$. Suppose there exists a subspace $B \subset X$ such that*

- $\bar{A} \subset B^\circ$,
- *$A \subset B$ is a deformation retract.*

Then the map $(X,A) \to (X/A, \{p\})$ where $p \in X$ is a map in $\mathbf{Top}_2$ and induces an isomorphism of H_n for all $n \in Z$, i.e.,

$$H_n(X,A) \cong H_n(X/A,\{p\}) \cong \begin{cases} H_n(X/A) & \text{if } n > 0, \\ H_0(X/A)/\mathbb{Z} & \text{if } n = 0. \end{cases}$$

Remark 2.4.2. The connecting map ∂ in Corollary 2.3.16 has a geometric interpretation whenever Theorem 2.4.1 applies and $n > 0$. In fact, for any $n > 0$, we get a sequence $H_n(A) \to H_n(X) \to H_n(X/A) \xrightarrow{\partial} H_{n-1}(A)$. Now suppose $\alpha \in S_n(X)$ such that $\partial\alpha$ is in the image of the inclusion $S_{n-1}(A) \hookrightarrow S_{n-1}(X)$. Let $\bar{\alpha}$ be the image of α in $S_n(X/A,\{p\})$. Since $\partial\bar{\alpha} = 0$, $\bar{\alpha} \in Z_n(X/A)$. In particular, $\bar{\alpha}$ represents a class in $H_n(X/A)$. On the other hand, $\partial\bar{\alpha}$ is computed by taking a lift of $\partial\alpha$ to $\widetilde{\partial\alpha} \in S_{n-1}(A)$. Observe that $\widetilde{\partial\alpha}$ is in fact in $Z_{n-1}(A)$, so it represents a class in $H_{n-1}(A)$. In any case, ∂ maps the equivalence class of $\bar{\alpha}$ to that of $\widetilde{\partial\alpha}$.

Before proving Theorem 2.4.1, let's apply it to compute the homology of n-spheres.

Theorem 2.4.3 (Homology of sphere). *For any $q \in \mathbb{N}$, we have*

$$H_n(\mathbb{S}^q) \cong \begin{cases} \mathbb{Z} & \text{for } n \in \{0,q\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.1}$$

Proof. As a simplification, we will prove (2.4.1) when $q \in \{1,2\}$ and leave for the reader the general case which follows from induction.

First, assume that $q = 1$. Consider the inclusion $\{0,1\} \hookrightarrow [0,1]$. Note that there is a long exact sequence

$$\begin{tikzcd}
{} \arrow[r, dotted] & H_n(\{0,1\}) \arrow[r] & H_n([0,1]) \arrow[r] & H_n([0,1],\{0,1\}) \arrow[dll, "\partial"] \\
& H_{n-1}(\{0,1\}) \arrow[r] & H_{n-1}([0,1]) \arrow[r] & H_{n-1}([0,1],\{0,1\}) \arrow[r, dotted] & {} .
\end{tikzcd}$$

Note that Theorem 2.4.1 implies that $H_n([0,1],\{0,1\}) \cong H_n(\mathbb{S}^1,\{p\})$. Then using (2.3.1) in the long exact sequence for all $n > 1$, we get exact sequences $0 \to H_n(\mathbb{S}^1) \to 0$. Therefore, $H_n(\mathbb{S}^1) \cong 0$ for $n > 1$. When $n = 1$ we have an exact sequence $0 \to H_1(\mathbb{S}^1) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z}$. Note that f sends $a\{0\} + b\{1\}$ to $a + b$, so $\ker f = \mathbb{Z}\{\{0\} - \{1\}\}$. Therefore, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$. Since $\mathbb{S}^1$ has one path component, $H_0(\mathbb{S}^1) \cong \mathbb{Z}$.

Second, assume that $q = 2$. Consider the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$. Then there is a long exact sequence

$$\begin{tikzcd}
{} \arrow[r, dotted] & H_n(\mathbb{S}^1) \arrow[r] & H_n(\mathbb{D}^2) \arrow[r] & H_n(\mathbb{D}^2,\mathbb{S}^1) \arrow[dll, "\partial"] \\
& H_{n-1}(\mathbb{S}^1) \arrow[r] & H_{n-1}(\mathbb{D}^2) \arrow[r] & H_{n-1}(\mathbb{D}^2,\mathbb{S}^1) \arrow[r, dotted] & {} .
\end{tikzcd}$$

Therefore, for any $n > 2$, we get an exact sequence $0 \to H_n(\mathbb{D}^2,\mathbb{S}^1) \to H_{n-1}(\mathbb{S}^1) = 0$. On the other hand, Theorem 2.4.1 implies that $H_n(\mathbb{D}^2,\mathbb{S}^1) = H_n(\mathbb{D}^2/\mathbb{S}^1) = H_n(\mathbb{S}^2)$. Therefore, $H_n(\mathbb{S}^2) \cong 0$ for $n > 2$. When $n = 2$, there is an exact sequence $0 = H_2(\mathbb{D}^2) \to H_2(\mathbb{S}^2) \to H_1(\mathbb{S}^1) = \mathbb{Z} \to H_1(\mathbb{S}^2) = 0$. Therefore $H_2(\mathbb{S}^2) \cong \mathbb{Z}$. Moreover, when $n = 1$, there is an exact sequence $0 = H_1(\mathbb{D}^2) \to H_1(\mathbb{S}^2) \to H_0(\mathbb{S}^1) \xrightarrow{f} H_0(\mathbb{D}^2)$ where f is an isomorphism. and the first group is also 0. Therefore, $H_1(\mathbb{S}^2) \cong 0$. Finally, since $\mathbb{S}^2$ has one path component, we have $H_0(\mathbb{S}^2) \cong \mathbb{Z}$. $\square$

Corollary 2.4.4. *$\mathbb{R}^a$ is not homeomorphic to $\mathbb{R}^b$ if $a \neq b$.*

Proof. Suppose not. Then the one-point compactifications of $\mathbb{R}^a$ and $\mathbb{R}^b$ would be homeomorphic. In other words, $\mathbb{S}^a \cong \mathbb{S}^b$. But $H_n(\mathbb{S}^a) \ncong H_n(\mathbb{S}^b)$ for $n = a$ or $n = b$. $\square$

Remark 2.4.5.
1. Along the line of Corollary 2.4.4, we can set up a theory of dimension using homology, but it is out of the scope of these notes.
2. Theorem 2.4.3 and Corollary 2.4.4 are examples of an answer to the Question 1.1.1.

Theorem 2.4.6 (Brouwer's fixed-point theorem). *For any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$, there exists a fixed point of f.*

Proof. Suppose there is no fixed point. Define $g : \mathbb{D}^2 \to \mathbb{S}^1$ as follows: join $f(x)$ and x and extend the line to the boundary point b in the direction of x. Set $g(x) := b$, see Figure 2.2. Note that $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2 \xrightarrow{g} \mathbb{S}^1$ is the identity map. Applying the homology functor H_1 to $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2 \xrightarrow{g} \mathbb{S}^1$ we get $\mathbb{Z} \to 0 \to \mathbb{Z}$ where the composition has to be identity which is a contradiction. $\square$

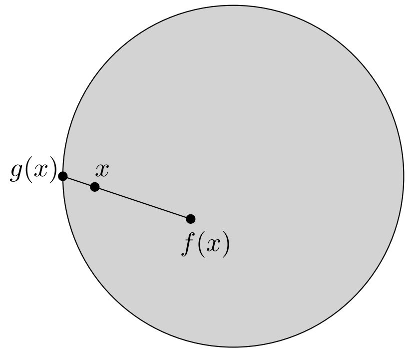


Figure 2.2: Brouwer's fixed-point theorem

In the rest of this section, we will prove that a general statement (cf. Theorem 2.4.8) implies Theorem 2.4.1. Since the proof of the statement is quite technical, we will postpone it to §2.5.

Definition 2.4.7. A triple $U \subset A \subset X$ of topological spaces is *excisive* if $\overline{U} \subset A^\circ$. An *excision* is the inclusion $(X-U, A-U) \hookrightarrow (X, A)$ an excisive triple.

Theorem 2.4.8 (Excision 2.0). *The excision of an excisive triple $U \subset A \subset X$ induces homology isomorphisms $H_m(X-U, A-U) \cong H_m(X, A)$.*

Remark 2.4.9. Theorem 2.4.8 says that the relative homology remains unaltered when we cut out spaces.

Proof of Theorem 2.4.1. Recall that the assumptions in Theorem 2.4.1 are $A \subset B \subset X$ is excisive, and $A \subset B$ is a deformation retract. Consider the following diagram in $\mathbf{Top}_2$,

$$\begin{tikzcd}
(X,A) \arrow[r, "i"] \arrow[d, "f"] & (X,B) \arrow[d] & (X-A, B-A) \arrow[l, "j"'] \arrow[d, "h"] \\
(X/A, \{p\}) \arrow[r, "\bar{i}"] & (X/A, B/A) & (X/A-\{p\}, B/A-\{p\}). \arrow[l, "\bar{j}"']
\end{tikzcd}$$

The idea is to prove that $i, \bar{i}, j, \bar{j}$ and h induce isomorphism in homology which would imply that f also induces an isomorphism.

Observe that h is an isomorphism in $\mathbf{Top}_2$, so it induces a homology isomorphism. Further, are excisions. Further, since $j, \bar{j}$ are excisions, Theorem 2.4.8 implies that j and $\bar{j}$ induce homology isomorphism. On the other hand, to prove that i induces a homology isomorphism, consider the diagram

$$\begin{tikzcd}
H_m(A) \arrow[r] \arrow[d, "i"] & H_m(X) \arrow[r] \arrow[d, "i"] & H_m(X,A) \arrow[r] \arrow[d, "i"] & H_{m-1}(A) \arrow[r] \arrow[d, "i"] & H_{m-1}(X) \arrow[d, "i"] \\
H_m(B) \arrow[r] & H_m(X) \arrow[r] & H_m(X,B) \arrow[r] & H_{m-1}(B) \arrow[r] & H_{m-1}(X)
\end{tikzcd}$$

The map $i : H_m(X, A) \to H_m(X, B)$ is an isomorphism by the Five lemma. Finally, to see that $\bar{i}$ induces a homology isomorphism it suffices to show that $\{p\} \to B/A$ is a homology isomorphism and use the Five lemma as before. However, a deformation retraction of B to A induces a deformation retraction from B/A to $\{p\}$. □

2.5 Locality principle

In this section, we will the *locality principle* that can be used to deduce Theorem 2.4.8.

Definition 2.5.1. 1. A *cover of* $X \in \mathbf{Top}$ is a family $\mathscr{A}$ of subsets of X such that $X = \bigcup_{A \in \mathscr{A}} A^\circ$.

2. An n-simplex $\sigma : \Delta^n \to X$ is $\mathscr{A}$*-small* if $\mathrm{Im}(\sigma) \subset A$ for some $A \in \mathscr{A}$.

Note that if σ is $\mathscr{A}$-small, so is $d_i\sigma$ for all i because $d_i\sigma : \Delta^{n-1} \subset \Delta^n \xrightarrow{\sigma} X$.

Definition 2.5.2. $\mathbf{Sing}^{\mathscr{A}}(X)$ is a sub semisimplicial set of $\mathbf{Sing}(X)$ containing the $\mathscr{A}$-small n-simplices and $S_*^{\mathscr{A}}$ is the chain complex associated to $\mathbf{Sing}^{\mathscr{A}}(X)$.

Theorem 2.5.3 (Locality Principle). *Suppose $\mathscr{A}$ is a cover of $X \in$ **Top**. Then the inclusion $S_*^{\mathscr{A}}(X) \subset S_*(X)$ is a homology isomorphism.*

The key idea in the proof Theorem 2.5.3 is that we can change any simplex $\sigma : \Delta^n \to X$ into a formal sum of $\mathscr{A}$-small simplicies by adding boundaries (cf. Theorem 2.5.7), i.e., we can choose the representative of $H_n(X)$ to be $\mathscr{A}$-small (Theorem 2.5.5 and Theorem 2.5.8). To formalize the idea, we construct a natural transformation \$ from S_n to itself which will turn an n-simplex into a formal sum of smaller n-simplices and does not change cycles modulo boundaries (cf. Theorem 2.5.5). Note that we must specify $S_n(\sigma)$ for all $\sigma : \Delta^n \to X$ to define \$. Further, the naturality of \$ implies that the following diagram commutes

$$\begin{tikzcd} S_n(\Delta^n) \arrow[r, "\$"] \arrow[d, "S_n(\sigma)"] & S_n(\Delta^n) \arrow[d, "S_n(\sigma)"] \\ S_n(X) \arrow[r, "\$"] & S_n(X). \end{tikzcd}$$

Observe $\mathbb{1}_{\Delta^n} \in S_n(\Delta^n)$ and $S_n(\sigma)(\mathbb{1}_{\Delta^n}) = \sigma \in S_n(X)$. Therefore, the above diagram implies that $\$(\sigma) = S_n(\sigma)(\$(\mathbb{1}_{\Delta^n}))$. In particular, \$ is determined by $\$(\mathbb{1}_{\Delta^n}) \in S_n(\Delta^n)$.[2] We will define $\$(\mathbb{1}_{\Delta^n})$ inductively.

For the base case, identify Δ^1 with $[0, 1]$ and write $\mathbb{1}_{\Delta^n}$ as $t \mapsto t$ for $t \in [0, 1]$. Define $f, g \in \mathbf{Sing}_1(\Delta^1)$ as $f(t) := \frac{t}{2}$ $g(t) := 1 - \frac{t}{2}$. Then $\$(\mathbb{1}_{\Delta^1}) := f - g$, see Figure 2.3.

[2] In fact, the class of natural transformations F from S_n to S_n are in bijection with the elements $F(\mathbb{1}_{\Delta^n}) \in S_n(\Delta^n)$. It is a special case of the *Yoneda Lemma.*

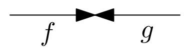


Figure 2.3: $\$(\mathbb{1}_{\Delta^n})$

In general, let b be the center of mass of Δ^n. Then define

$$\$(\mathbb{1}_{\Delta^n}) \coloneqq b * \$(\partial \mathbb{1}_{\Delta^n}) \tag{2.5.1}$$

where $b* : S_{n-1}(\Delta^n) \to S_n(\Delta^n)$ is the *cone construction* (2.1.2) which takes $\sigma : \Delta^{n-1} \to \Delta^n$ to $b * \sigma(t_0, \dots, t_n) \coloneqq t_0 b + (1 - t_0)\sigma(\frac{(t_1,\dots,t_n)}{1-t_0})$.

Geometrically, \$ breaks a standard n-simplex into small n-simplices by joining its barycenter with the mid-points of its faces. To visualize the division in a simple case note that $\$(\mathbb{1}_{\Delta^2}) = A - B + C - D + E - F$, see Figure 2.4.

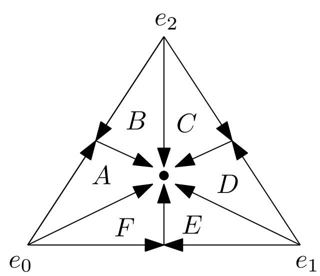


Figure 2.4: Barycenteric division

Lemma 2.5.4. *The maps* $\$: S_n(X) \to S_n(X)$ *induce a chain map* $\$: S_*(X) \to S_*(X)$.

Proof. By naturality, it suffices to check that $\partial \$ \mathbb{1}_{\Delta^n} = \$ \partial \mathbb{1}_{\Delta^n}$. Check that this holds inductively. One has to make use of the fact that $b*$ is a chain homotopy between the identity and $\eta\epsilon$, see (2.1.2). □

Theorem 2.5.5. *The chain map* $\$: S_*(X) \to S_*(X)$ *is naturally chain homotopic to* $\mathbb{1}_{S_*(X)}$

Proof. By naturality, it suffices to define $T(\mathbb{1}_{\Delta^n}) \in S_{n+1}(\Delta^n)$ to construct a natural chain homotopy $T : S_n \to S_{n+1}$. As before, let b be the center of mass of Δ^n. Define inductively

$$T(\mathbb{1}_{\Delta^n}) \coloneqq b * (\$\mathbb{1}_{\Delta^n} - \mathbb{1}_{\Delta^n} - T(\partial \mathbb{1}_{\Delta^n})), \tag{2.5.2}$$

where $T(\mathbb{1}_{\Delta^0}) = 0$. Check inductively that this map is a chain homotopy. By universality, we need to check that $\partial T(\mathbb{1}_{\Delta^n}) + \partial T(\mathbb{1}_{\Delta^n}) = \$(\mathbb{1}_{\Delta^n}) - \mathbb{1}_{\Delta^n}$. □

Remark 2.5.6. Geometrically, $T(\mathbb{1}_{\Delta^1}) \in S_2(\Delta^1)$ will be a generator $T(\mathbb{1}_{\Delta^1}) : \Delta^2 \to \Delta^1$ that squashes the triangle Δ^2 to a line Δ^1.

Theorem 2.5.7. *Suppose* $X \in \mathbf{Top}$ *and* $\mathscr{A}$ *is a cover of* X. *For any* $\sigma \in \mathbf{Sing}_n(X)$ *there exists* $k \geq 0$ *such that* $\$^k(\sigma) \in S_n(X)$ *is* $\mathscr{A}$ *small where* $\$^k \coloneqq \underbrace{\$ \circ \cdots \circ \$}_{k \text{ times}}$.

Proof. First, consider the special case $X = \Delta^n$ and $\sigma = \mathbb{1}_{\Delta^n}$. Then Theorem 2.5.7 is a consequence of the Lebesgue covering lemma which says that if $\mathscr{U}$ is a cover of Δ^n then there exists $\epsilon > 0$ such that for all $x \in \Delta^n$, $B(x, \epsilon) \subset U$ for some $U \in \mathscr{U}$. In general, note that $\mathscr{U} := \{\sigma^{-1}(A) \mid A \in \mathscr{A}\}$ is a cover of Δ^n. If we choose k such that $\$^k(\mathbb{1}_{\Delta^1})$ is $\mathscr{U}$ small then by naturality $\$^k(\sigma)$ will be $\mathscr{A}$ small. ☐

Theorem 2.5.8. *For any $k \geq 1$ and any $X \in$ **Top**, the chain map $\$^k : S_*(X) \to S_*(X)$ is naturally chain homotopic to $\mathbb{1}_{S_*(X)}$. Explicitly, there exist natural transformations $T_{k,m} : S_m(X) \to S_{m+1}(X)$ such that $\partial T_{k,m} + T_{k,m-1}\partial = \$^k - \mathbb{1}$.*

Proof. Iterate the argument in Theorem 2.5.5. ☐

Lemma 2.5.9. *If $\sigma \in S_n(X)$ is $\mathscr{A}$ small, so is $T_k\sigma$.*

Proof. Observe that σ is the composite $\Delta^n \xrightarrow{\bar{\sigma}} A \xhookrightarrow{i} X$ for some $A \in \mathscr{A}$. Then by naturality of T_k we have $T_k\sigma = (S_{m+1}(i))(T_k\bar{\sigma})$, i.e., $T_k\sigma$ is the image of $T_k\bar{\sigma} \in S_{m+1}(A)$. Therefore, $T_k\sigma$ is $\mathscr{A}$ small. ☐

Proof of Theorem 2.5.3. It suffices to show that there exists a map for every $m \in \mathbb{Z}$ such that $H_m^{\mathscr{A}}(X) \to H_m(X)$ is an isomorphism in **Ab**.

- **Surjectivity:** Suppose $\alpha \in Z_m(X)$. Pick k such that $\$^k\alpha \in \mathrm{Im}(S_m^{\mathscr{A}}(X) \to S_m(X))$ is $\mathscr{A}$-small. Note that $\partial(\$\alpha) = \$(\partial\alpha) = \$(0) = 0$, so $\$\alpha$ is a cycle, and by iteration so is $\$^k\alpha$. Now observe that $\$^k\alpha - \alpha = \partial T_{k,m}\alpha + T_{k,m}\partial\alpha = \partial T_{k,m}\alpha$ which is a boundary. In particular, α is $\mathscr{A}$-small modulo a boundary which implies surjectivity.

- **Injectivity:** Suppose $\alpha \in Z_m^{\mathscr{A}}(X)$ has trivial image in $H_m(X)$, i.e., $\alpha = \partial\beta$ for some $\beta \in S_{m+1}(X)$. To prove injectivity, it suffices to show that we can choose β to be $\mathscr{A}$-small. To this end, choose k such that $\$^k\beta$ is $\mathscr{A}$ small. Then

$$\partial(\$^k\beta) - \alpha = \partial(\$^k - \mathbb{1})\beta = \partial(\partial T_{k,m} + T_{k,m}\partial)\beta = \partial T_k \partial\beta = \partial T_k\alpha.$$

  Note that Lemma 2.5.9 implies that $T_k\alpha$ is $\mathscr{A}$-small. Therefore, α is an $\mathscr{A}$-small boundary.

☐

Proof of Theorem 2.4.8. Suppose $U \subset A \subset X$ is an excisive triple, i.e., $\bar{U} \subset A^\circ$. Define a cover $\mathscr{A} := \{A, X - U\}$ of X. Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r, "i"] \arrow[d, "f"] & S_*^{\mathscr{A}}(X) \arrow[r, "j"] \arrow[d, "g"] & S_*^{\mathscr{A}}(X)/S_*(A) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X, A) \arrow[r] & 0,
\end{tikzcd}$$

where f is an isomorphism and g is a homology isomorphism given in the proof of Theorem 2.5.3, h is induced by inclusion, i is an inclusion, and j is projection. Further, $S_*^{\mathscr{A}}(X, A) = S_*^{\mathscr{A}}(X)/S_*(A)$. Therefore, we have a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & H_m(S_*(A)) \arrow[r] \arrow[d, "H_m(f)"] & H_m(S_*^{\mathscr{A}}(X)) \arrow[r] \arrow[d, "H_m(g)"] & H_m(S_*^{\mathscr{A}}(X)/S_*(A)) \arrow[d, "H_m(h)"] \arrow[dll, rounded corners] \\
\cdots \arrow[r, dotted] & H_m(S_*(A)) \arrow[r] & H_m(S_*(X)) \arrow[r] & H_m(S_*(X, A)) \arrow[dll, rounded corners] \\
& H_{m-1}(S_*(A)) \arrow[r] \arrow[d, "H_{m-1}(f)"] & H_{m-1}(S_*^{\mathscr{A}}(X)) \arrow[r] \arrow[d, "H_{m-1(g)}"] & H_{m-1}(S_*^{\mathscr{A}}(X)/S_*(A)) \arrow[d, "H_{m-1}(h)"] \arrow[r, dotted] & \cdots \\
& H_{m-1}(S_*(A)) \arrow[r] & H_{m-1}(S_*(X)) \arrow[r] & H_{m-1}(S_*(X, A)) \arrow[r, dotted] & \cdots
\end{tikzcd}.$$

Note that $H_m(f)$, $H_m(g)$, $H_{m-1}(f)$ and $H_{m-1}(f)$ are isomorphisms. Therefore using the Five Lemma 2.3.12 implies that $H_m(h)$ is a homology isomorphism. In particular, $h : S_*^{\mathscr{A}}/S_*(A) \to S_*(X)/S_*(A)$ is a homology isomorphism.

On the other hand, if σ is $\mathscr{A}$-small, $\operatorname{Im}\sigma$ is either contained in A or $X - U$ which implies that

$$S_l^{\mathscr{A}}(X)/S_l(A) = \frac{S_l(A) + S_l(X-U)}{S_l(A)} \cong \frac{S_l(X-U)}{S_l(A) \cap S_l(X-U)} \cong \frac{S_l(X-U)}{S_l(A-U)}.$$

where we used one of the isomorphism theorems in group theory in the second equality. To summarize, we have $H_m(X, A) \cong H_m(X - U, A - U)$. □

2.6 Mayer–Vietoris

In the last couple of sections, we completed proof of the excision theorem using the locality principle. However, the main goal for us is to compute homology putting the proofs of the axioms under the rug. In this section, we will prove Mayer–Vietoris theorem 2.6.1 as an application of the locality principle. The theorem will help us to compute the homology of a space X by looking at the homology of elements of a cover of X.

Theorem 2.6.1 (Mayer–Vietoris). *Consider a topological space X with a cover $\mathscr{A} = \{A, B\}$. Then there is a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & H_n(A \cap B) \arrow[r, "H_n(i) \oplus H_n(j)"] & H_n(A) \oplus H_n(B) \arrow[r, "H_n(k) - H_n(l)"] & H_n(X) \arrow[dll, rounded corners, "\partial"] \\
& H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \arrow[r, dotted] & \cdots
\end{tikzcd},$$

where $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$ and $l : B \hookrightarrow X$ are inclusion maps.

Proof. Observe that $0 \to S_*(A \cap B) \to S_*(A) \oplus S_*(B) \to S_*^{\mathscr{A}}(X) \to 0$ is a short exact sequence of chain complexes. Therefore, Theorem 2.3.15 implies that there is a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, dotted] & H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[r] & H_n^{\mathscr{A}}(X) \arrow[dll, "\partial", rounded corners] \\ & H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & H_{n-1}^{\mathscr{A}}(X) \arrow[r, dotted] & \cdots \end{tikzcd}$$

Finally, observe that Theorem 2.5.3 implies that $H_n^{\mathscr{A}}(X) = H_n(X)$. □

Theorem 2.6.2. *For a collection $\{X_i \in \mathbf{Top}\}_{i \in I}$ we have $H_n(\sqcup_{i \in I} X_i) \cong \bigoplus_{i \in I} H_n(X_i)$.*

In the rest of this section, we will use Mayer–Vietoris to re-compute the homology of $\mathbb{S}^1$ and $\mathbb{S}^2$ (see Theorem 2.4.3). Further, we will compute the homology of a torus $\mathbb{T}^2$.

Theorem 2.6.3 (Homology of sphere revisited). *For any $q \in \mathbb{N}$, we have*

$$H_n(\mathbb{S}^q) \cong \begin{cases} \mathbb{Z} & \text{for } n \in \{0, q\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.1}$$

Proof. We will compute $H_1(\mathbb{S}^1)$ and $H_2(\mathbb{S}^2)$. For the first case, parametrize $\mathbb{S}^1$ by $\theta \in [0, 2\pi)$. Define $A := \mathbb{S}^1 \setminus \{\theta = 0\}$ and $B := \mathbb{S}^1 \setminus \{\theta = -\pi\}$. Note that $H_1(A \cap B) \cong 0$ and $H_1(A) \oplus H_1(B) \cong 0$. Further, $H_0(A \cap B) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(A) \oplus H_0(B) \cong \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, Theorem 2.6.1 implies that there is a sequence $0 \to H_1(\mathbb{S}^1) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}$. Note that f sends $(a, b) \mapsto (a + b, a + b)$. Therefore, $\ker f = \{(a, -a)\} \cong \mathbb{Z}$, i.e., $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

For the second case, define $A := \mathbb{S}^2 \setminus \{e_1\}$ and $B := \mathbb{S}^2 \setminus \{-e_1\}$ and $H_1(A) \oplus H_1(B) \cong 0$. Note that $H_2(A) \cong H_2(B) \cong 0$. Further, $H_1(A \cap B) \cong \mathbb{Z}$ since $A \cap B$ can be deformed to a circle. Therefore, Theorem 2.6.1 implies that $H_2(\mathbb{S}^2) \cong \mathbb{Z}$. □

Theorem 2.6.4 (Homology of torus). *Define $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Then*

$$H_m(\mathbb{T}^2) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } m = 1, \\ \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose A_1 and A_2 are the left and right parts of the torus and B_1, B_2 are the cylinders in $A_1 \cap A_2$, see Figure 2.5. Note that for $m \geq 3$, $H_m(A_i) \cong 0$ and for $m \geq 2$ $H_m(A_1 \cap A_2) \cong 0$. Therefore, Theorem 2.6.1 implies that $H_m(\mathbb{T}^2) \cong 0$ for $m \geq 3$.

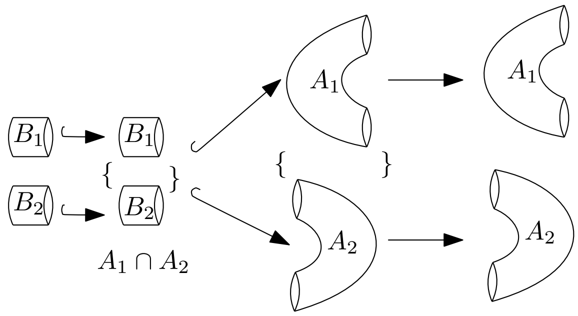


Figure 2.5: Mayer–Vietoris for a torus

On the other hand, $H_1(A_i) \cong \mathbb{Z}$, and $H_1(A_1 \cap A_2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, Theorem 2.6.1 implies that there is an exact sequence $0 \to H_2(\mathbb{T}^2) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}$. To compute $H_2(\mathbb{T}^2)$ it suffices to understand $f : H_1(A_1 \cap A_2) \to H_1(A_1) \oplus H_1(A_2)$. Consider the map

$$B_i \hookrightarrow A_1 \cap A_2 \hookrightarrow A_1 \sqcup A_2 \to A_i \tag{2.6.2}$$

where the first two are inclusions and the last one is projection, see Figure 2.5. The maps induce maps of homology groups $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ where the composition has to be identity, i.e., $1 \mapsto (1,0) \mapsto f(1,0) \mapsto 1$ and $1 \mapsto (0,1) \mapsto f(0,1) \mapsto 1$. It implies that f maps $(a,b) \mapsto (a+b, a+b)$. Hence, $\ker f \cong \mathbb{Z}$ and $H_2(\mathbb{T}^2) \cong \mathbb{Z}$.

Similarly, one can check that $H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

Remark 2.6.5. There is a subtle sign issue with the construction of f in Theorem 2.6.4. Namely, the isomorphism induced by the maps in (2.6.2) can send a generator (anticlockwise loop) of homology to ± 1. However, once we fix a basis the isomorphism is determined and the proof works. Later, we will develop a homology theory where we don't have to deal with a choice of basis (cf. Remark 4.1.6 2).

2.7 Summary of homology

Before proceeding further, let's record the properties of homology that we proved so far.

1. There is a sequence of functors $H_n : \mathbf{Top}_2 \to \mathbf{Ab}$, $n \in \mathbb{Z}$.
2. There are natural transformations $\partial : H_n(X, A) \to H_{n-1}(A, \emptyset)$ of functors $\mathbf{Top}_2 \to \mathbf{Ab}$ such that the following holds true:
   (a) **Long exact sequence:** For any pair $(X, A) \in \mathbf{Top}_2$, the following is a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, dotted] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, rounded corners, "\partial"] \\ & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X,A) \arrow[r, dotted] & \cdots \end{tikzcd}$$

(b) **Homotopy invariance:** If $f_0, f_1 : (X, A) \to (X, B)$ are homotopic, they induce the same maps $H_n(*)$.

(c) **Excision:** If $U \subset A \subset X$ is excisive, then the inclusion $(X - A, A - U) \hookrightarrow (X, A)$ induces a homology isomorphism for all $n \in \mathbb{Z}$.

(d) **Coproduct:** $H_n(\sqcup_{i \in I} X_i) \cong \bigoplus_{i \in I} H_n(X_i)$.

(e) **Dimension axiom:** If p is a point then

$$H_m(\{p\}, \emptyset) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The points listed above are called the *Eilenberg–Steenrod axioms* for homology. They determine the homology functors on all reasonably nice spaces that appear in manifold theory, combinatorics, and algebra.

The fact that there are non-homotopic spaces (cf. Remark 5.3.10) that have the same homology ignited a search for more invariants that can distinguish spaces up to homotopy equivalence. Extraordinay homology, a topic that is out of the scope of these notes, is an example of another invariant.

An *extraordinary homology* is a sequence of functors E_n and natural transformations ∂ satisfying all the Eilenberg–Steenrod axioms but the dimension axiom. Some examples of extraordinary homology theory are

- K-theory,
- Bordism,
- Topological modular forms, and
- Morava E-theories.

Chapter 3

Computing homology II

The homology theory we have developed so far requires us to understand $S_*(X)$ and the boundary maps if we were to carry out any computation. However, $S_n(X) = \mathbb{Z}\{C(\Delta^n; X)\}$ being infinitely generated can be quite hard to understand. On the other hand, the algorithm to compute homology we alluded to in §1.1 bypassed the necessity to understand $C(\Delta^n; X)$. Prioritizing the computational aspect, we will introduce in §3.1 a combinatorial object CW-complex by triangulating a space. Then in §3.3 we will show that the homology of the triangulation is the same as the singular homology. As an application, we will study in §3.4 how to distinguish maps between spheres and in §3.5 we will compute the homology of real projective space. Finally, in 3.6 we will transition into studying topological invariants other than homology that will help us answer the Question 1.1.1.

3.1 CW complexes

CW complexes provide finite triangulations on $X \in \mathbf{Top}$ for which the computation boils down to combinatorics. Further, the process of triangulation is continuous in some sense. In this section, we will frame CW complexes in a categorical language.

Definition 3.1.1. Given a diagram in a category $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C, & \end{tikzcd}$$

the *pushout* is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "p_B"] \\ C \arrow[r, "p_C"] & p \end{tikzcd}$$

such that if any other object z of $\mathcal{C}$ satisfies the above commutative diagram, there exists a unique map $p \to z$ such that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & p \arrow[dr, dotted] & \\ & & z. \end{tikzcd}$$

Example 3.1.2. 1. The pushout of a diagram in **Top**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

is $p \coloneqq B \sqcup C/(f(a) \sim g(a)$ for all $a \in A)$. In particular, if $A = \emptyset$ then $p = B \sqcup C$. When C is a point and A is general then the pushout is $B/\operatorname{Im} f$.

Definition 3.1.3. We say that a pushout

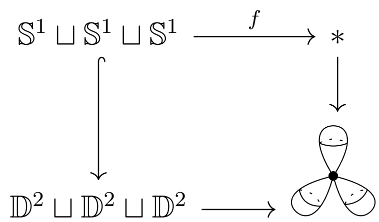
$$\begin{tikzcd} \sqcup_{i\in I}\mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & B \arrow[d] \\ \sqcup_{i\in I}\mathbb{D}^n \arrow[r] & p \end{tikzcd}$$

is *obtained from B by attaching n-cells, $\mathbb{D}^n$, along their boundaries* $\sqcup_{i\in I}\mathbb{S}^{n-1}$. The attaching map is determined by f.

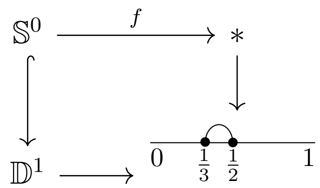
Example 3.1.4. 1. The figure 8 is obtained from a point by attaching two 1-cells. In particular, we have a diagram

$$\begin{tikzcd} \mathbb{S}^0 \sqcup \mathbb{S}^0 \arrow[r, "f"] \arrow[d, hook] & * \arrow[d] \\ \mathbb{D}^1 \sqcup \mathbb{D}^1 \arrow[r] & 8. \end{tikzcd}$$

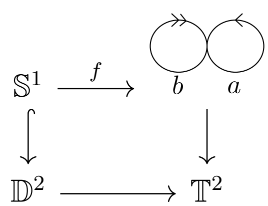
2. If we attach three 2-cells to a point, we get the following pushout:



3. Define a map from $\mathbb{S}^0 = \{x, y\} \hookrightarrow \mathbb{D}^1$ such that $x \mapsto 1/3$ and $y \mapsto 1/2$. If $\mathbb{S}^0$ is attached to a point, then we get the following pushout diagram where $1/2$ is identified with $1/3$.



4. We can think of a torus as a pushout



where we consider $\mathbb{S}^1$ as a square with sides a and b, see Figure 3.1 and f glues $\mathbb{S}^1$ to the figure 8 along $aba^{-1}b^{-1}$. Check pictorially that the pushout is the same as $\mathbb{S}^1 \times \mathbb{S}^1$.

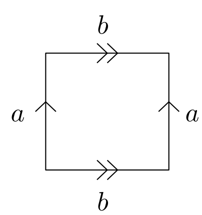


Figure 3.1: Boundary of attaching disc

In general, we consider the model spaces in **Top** as spaces formed by gluing n-cells which is made precise in the following definition. As we will see in the next section, the homology theory of these model spaces is combinatorial in nature.

Definition 3.1.5. A *CW-complex*[1] is a space $X \in$ **Top** together with a filtration by subspaces

$$\emptyset = Sk_{-1}(X) \subset Sk_0(X) \subset Sk_1(X) \subset \cdots \subset X$$

such that the *skeleta* $Sk_i(X)$ satisfy the following properties:

[1] Here C stands for cell and W for weak topology, i.e., $A \subset X$ is open if and only if the restriction of A to every skeleton is open.

- $X = \bigcup_i Sk_i(X)$ and
- there are pushouts

$$\begin{tikzcd} \sqcup_{i\in I_n}\mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & Sk_{n-1}(X) \arrow[d] \\ \sqcup_{i\in I_n}\mathbb{D}^{n} \arrow[r] & Sk_n(X). \end{tikzcd}$$

Remark 3.1.6. 1. All CW complexes are Hausdorff.

2. Models of topological spaces like smooth manifolds and algebraic varieties can always be equipped with CW structures.

Example 3.1.7. 1. A torus has a CW complex structure defined by $Sk_{-1}\mathbb{T}^2 := 0, Sk_0\mathbb{T}^2 := *, Sk_1\mathbb{T}^2 :=$ Figure 8, and $Sk_i\mathbb{T}^2 := \mathbb{T}^2$ for all $i \geq 2$.

2. $\mathbb{S}^n$ has a CW complex structure with one 0-cell and one n-cell . For instance, the structure on $\mathbb{S}^2$ is $\emptyset \subset * \subset * \subset \mathbb{S}^2 \subset \mathbb{S}^2 \ldots$.

3. There is another CW structure on $\mathbb{S}^2$ with $Sk_0\mathbb{S}^2 = \mathbb{S}^0$, $Sk_1\mathbb{S}^2 = \mathbb{S}^1$ and $Sk_i\mathbb{S}^2 = \mathbb{S}^2$ for $i \geq 2$. Geometrically, we start with two 0-cells, attach them to two 1-cells along the boundaries, see Figure 3.2. Then add two 2-cells as upper and lower hemispheres.

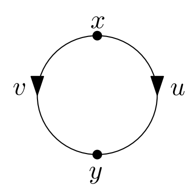


Figure 3.2: $Sk_1(\mathbb{S}^2)$

Definition 3.1.8. 1. A CW complex X is *finite dimensional* if $Sk_n(X) = X$ for some finite n. The minimum n for which $Sk_n(X) = X$ is called the *dimension of* X. If the dimension is not finite, we say that X is *infinite-dimensional.*

2. A CW complex is *finite* if it is finite-dimensional with finitely many cells in each dimension.

Remark 3.1.9. Finite CW complexes are compact.

Example 3.1.10. 1. $\mathbb{S}^2$ is a two-dimensional CW complex.

2. There is an infinite dimensional CW complex $\mathbb{S}^\infty$ with $Sk_i\mathbb{S}^\infty = \mathbb{S}^i$. Recall that, for any $q \in \mathbb{N}$,

$$H_n(\mathbb{S}^q) \cong \begin{cases} \mathbb{Z} & \text{for } n \in \{0, q\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

In particular, as $n \to \infty$, we should expect that $H_i(\mathbb{S}^\infty) = 0$ for all i and $H_0(\mathbb{S}^\infty) = \mathbb{Z}$. Therefore, $\mathbb{S}^\infty$ is close to being a point. In fact, the following proposition implies that they are homotopy equivalent.

Proposition 3.1.11. $\mathbb{S}^\infty \simeq *$.

Proof. Observe that $\mathbb{S}^\infty = \{(x_0, x_1, \dots) \mid \sum x_i^2 = 1, x_i = 0 \text{ for all } i \geq N \text{ for some } N > 0\}$. Consider the maps $f : \mathbb{S}^\infty \to *$ and $g : * \to \mathbb{S}^\infty$. Note that fg is identity. We just need to prove that $gf \cong \mathbb{1}_{\mathbb{S}^\infty}$.

First, define a right-shift operator[2] $T : \mathbb{S}^\infty \to \mathbb{S}^\infty$ such that $(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$. Further, define $h : \mathbb{S}^\infty \times [0,1] \to \mathbb{S}^\infty$ as

$$h(x,t) := \frac{tx + (1-t)Tx}{\|tx + (1-t)Tx\|}.$$

Note that the denominator is non-zero unless $t = 0$ in which case $\|Tx\| = 1$. Therefore, h is a well-defined homotopy from T to $\mathbb{1}_{\mathbb{S}^\infty}$. Similarly,

$$h'(x,t) := \frac{t(1,0,\dots) + (1-t)Tx}{\|t(1,0,\dots) + (1-t)Tx\|}.$$

is a homotopy between a constant function and T. Then the concatenation of h and h' is the required homotopy $gf \cong \mathbb{1}_{\mathbb{S}^\infty}$. □

Definition 3.1.12. The *real projective space* $\mathbb{RP}^n$ is $\mathbb{S}^n / x \sim -x$.

Note that $\mathbb{RP}^0$ is a point and $\mathbb{RP}^1 = \mathbb{S}^1$. Further, $\mathbb{RP}^2$ has a CW structure with one of each 0-cell, 1-cell, and 2-cell, i.e., $Sk_0\mathbb{RP}^2 = \mathbb{RP}^0 \subset Sk_1 = \mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \dots$. In general, $\mathbb{RP}^n$ has CW structure with $Sk_i\mathbb{RP}^n = \mathbb{RP}^i$ for $i \leq n$. We can also define a CW complex $\mathbb{RP}^\infty := \cup \mathbb{RP}^i$. Unlike $\mathbb{S}^\infty$, $\mathbb{RP}^\infty$ has different homology than that of a point. In fact, the homology groups of $\mathbb{RP}^\infty$ have torsion (cf. Theorem 3.5.1).

Definition 3.1.13. A map $f : X \to Y$ of CW complexes is *cellular* if the image $f(Sk_i(X)) \subset Sk_i(Y)$ for all i.

Example 3.1.14. There is a cellular map $\mathbb{S}^1 \hookrightarrow \mathbb{S}^2$ if we equip them with CW structures.

Remark 3.1.15.
1. There is a category **CWcomp** of CW complexes and cellular maps.
2. There is a functor called *geometric realization* from the category of semisimplicial sets to the category of cell complexes, $\mathbf{Fun}(\Delta_{inj}^{op}, \mathbf{Set}) \to \mathbf{CWcomp}$. If X_* is a semisimplicial set, then the geometric realization of X_* is $(\coprod_n X_n \times \Delta^n)/\sim$ where we "identify the boundary" under the equivalence relation. Intuitively, we can think of X_0 as vertices, X_1 as edges, and so on, glued together using boundary maps.
3. The geometric realization functor allows us to infer properties of semisimplicial sets by studying CW complexes. In particular, we can reduce the study of a wildly infinite group $S_n(X)$ to possibly finite combinatorial object, CW complex.

[2]The shift operator in infinite dimensional topology is a simple case of *swindles*.

3.2 Homology of skeleta

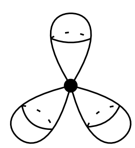
The building blocks of a CW complex are its skeleta. In this section, we will study the homology of skeleta and use them in upcoming sections to compute the homology of the CW complex. In particular, we will answer the following question:

Question 3.2.1. Given a CW complex X, what is the relation between $H_q(Sk_n X)$, $H_q(Sk_{n-1} X)$ and $H_q(X)$?

Definition 3.2.2. Given an index set I, *a wedge of n-spheres* is the space

$$\bigvee_{i \in I} \mathbb{S}^n := \sqcup_{i \in I_n} \mathbb{S}^n / \sqcup_{i \in I} *.$$

A wedge of n-spheres is a collection of n-spheres glued together at a point $*$. For instance, when $n = 2$, $\bigvee_{i \in \{a,b,c\}} \mathbb{S}^2$ is



The wedge of n-spheres is fundamental since for any CW complex X we have

$$Sk_n(X)/Sk_{n-1}(X) \cong \bigvee_{i \in I_n} \mathbb{S}^n. \tag{3.2.1}$$

In fact, there is a pushout diagram

$$\begin{tikzcd}
\sqcup_{i \in I_n} \mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & Sk_{n-1}(X) \arrow[d] \\
\sqcup_{i \in I_n} \mathbb{D}^n \arrow[r] & Sk_n(X)
\end{tikzcd}$$

which means we glue a collection of disks to Sk_{n-1} along ther boundaries. Therefore, when we collapse the boundaries of the discs, we get n-spheres wedged at a point.

On the other hand, there is a long exact sequence associated to the inclusion $Sk_{n-1} X \subset Sk_n X$:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & H_q(Sk_{n-1}X) \arrow[r] & H_q(Sk_n X) \arrow[r] & H_q(Sk_n X, Sk_{n-1}X) \arrow[dll, "\partial"] \\
& H_{q-1}(Sk_{n-1}X) \arrow[r] & H_{q-1}(Sk_n X) \arrow[r] & H_{q-1}(Sk_n X, Sk_{n-1}X) \arrow[r, dotted] & \cdots
\end{tikzcd}.$$

Note that an $n-1$-sphere is a deformation retract of a pinched n-disk. Similarly, $Sk_{n-1}(X)$ is a deformation retract of $Sk_nX \setminus \{p\}$. Thus $Sk_{n-1}(X) \subset Sk_n(X)$ is an excision which implies $H_q(Sk_n(X), Sk_{n-1}(X)) \cong \tilde{H}_q(Sk_n(X)/Sk_{n-1}(X)) \cong \tilde{H}_q(\bigvee_{i\in I_n} \mathbb{S}^n)$ where $\tilde{H}_q(Y)$ is the *reduced homology* $H_q(Y,*)$. Therefore, to answer the Question 3.2.1, we just need to compute $H_q(\bigvee_{i\in I_n} \mathbb{S}^n)$.

To this end, there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & H_q(\sqcup *) \arrow[r] & H_q(\sqcup \mathbb{S}^n) \arrow[r] & H_q(\sqcup \mathbb{S}^n, \sqcup *) \arrow[dll, rounded corners, "\partial"'] & \\
& H_{q-1}(\sqcup *) \arrow[r] & H_{q-1}(\sqcup \mathbb{S}^n) \arrow[r] & H_{q-1}(\sqcup \mathbb{S}^n, \sqcup *) \arrow[r, dotted] & \cdots
\end{tikzcd}$$

which implies

$$H_q(\sqcup_{i\in I_n}\mathbb{S}^n, \sqcup_{i\in I_n} *) \cong H_q(\bigvee_{i\in I_n} \mathbb{S}^n, *) \cong \begin{cases} 0 & \text{if } q = 0, \\ H_q(\bigvee_{i\in I_n} \mathbb{S}^n) & \text{if } q > 0. \end{cases}$$

Therefore,

$$H_q(\bigvee_{i\in I_n} \mathbb{S}^n, *) \cong \begin{cases} \oplus_{i\in I_n}\mathbb{Z} & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

To summarize, we have a partial answer to the Question 3.2.1:

Theorem 3.2.3. *Suppose X is a CW complex. Then*

1. *$H_q(Sk_{n-1}X) = H_q(Sk_nX)$ unless $q = n$ or $q = n-1$ which implies that $H_q(Sk_nX) = 0$ for $n < q$, and*
2. *homology stabilizes, i.e., $H_q(X) = H_q(Sk_nX)$ for $n > q$.*

Just for the record, we can define the wedge sum of two pointed spaces $(X, x), (Y, y)$ as $X \vee Y := X \sqcup Y/(x \sim y)$. Then the same argument given above implies the following theorem.

Theorem 3.2.4. *For any $X, Y \in$ **Top** we have*

$$H_n(X \vee Y) = \begin{cases} H_n(X \sqcup Y) - \mathbb{Z} & \text{if } n = 0, \\ H_n(X \sqcup Y) & \text{if } n > 0. \end{cases} \tag{3.2.2}$$

3.3 Cellular homology

The case when $q = n$ or $n-1$ motivates further investigation. In this section, we will define a (*cellular*) chain complex using the homology groups for these edge cases and prove that we can recover the homology of X from this chain complex (cf. Theorem 3.3.2). The advantage of the cellular chain complex is that it is combinatorial.

Definition 3.3.1. The *cellular chain complex* $C_*^{cell}(X)$ *of a CW complex* X has n^{th}-entry

$$C_n^{cell}(X) := H_n(Sk_n(X), Sk_{n-1}(X)) \cong \oplus_{i \in I_n} \mathbb{Z}, \tag{3.3.1}$$

where the isomorphism depends on a choice of pushout. Further, a boundary map $d : C_n^{cell}(X) \to C_{n-1}^{cell}(X)$ is the composition

$$\begin{tikzcd}
C_n^{cell}(X) = H_n(Sk_n(X), Sk_{n-1}(X)) \arrow[r, "\partial"] & H_{n-1}(Sk_{n-1}(X)) \arrow[d] \\
& H_{n-1}(Sk_{n-1}(X), Sk_{n-2}(X)) = C_{n-1}^{cell}(X)
\end{tikzcd}$$

where ∂ is the connecting map in the Snake Lemma 2.3.15 and the other map is induced by the quotient map.

Theorem 3.3.2. *$(C_*^{cell}(X), d)$ is a well-defined chain complex. Writing H_*^{cell}, Z_*^{cell} and B_*^{cell} to mean the homology, cycles, and boundaries of (C_*^{cell}, d) respectively, we have*

$$H_n^{cell}(X) \cong H_n(X). \tag{3.3.2}$$

Remark 3.3.3.

1. (C_*^{cell}, d) is a functor $\mathbf{CWcomp} \to \mathbf{chAb}$.
2. In general, the chain complex $C_*^{cell}(X)$ is extremely smaller than the chain complex $S_*(X)$. For instance, $C_n^{cell}(X)$ is usually a direct sum of copies of $\mathbb{Z}$ as we see in the examples below.
3. In practice, $d : C_n^{cell}(X) \to C_{n-1}^{cell}(X)$ is calculated by composing of the following maps:

$$\sqcup_{i \in I_n} \mathbb{S}^{n-1} \to Sk_{n-1}(X) \to Sk_{n-1}(X)/Sk_{n-2}(X) = \bigvee_{i \in I_{n-1}} \mathbb{S}^{n-1}.$$

For a given n-cell, the first map says where its boundary goes in $Sk_{n-1}(X)$. Then we pinch $Sk_{n-2}(X)$ to get the resulting element in $Sk_{n-1}(X)/Sk_{n-2}(X)$.

Before proving the result, let's recover the homologies of $\mathbb{S}^n$ and $\mathbb{T}^2$ using cellular homology.

Example 3.3.4. 1. $\mathbb{S}^n$ has a CW structure with one 0-cell and one n-cell. Therefore, $(C_*^{cell}(\mathbb{S}^n), d)$ is

$$\dashrightarrow 0 \to C_n^{cell} = \mathbb{Z} \to 0 \dashrightarrow 0 \to C_0^{cell} = \mathbb{Z} \to 0. \tag{3.3.3}$$

Therefore, $H_q(\mathbb{S}^n) = \mathbb{Z}$ for $q = 0, n$ and 0 otherwise.

Check that we get the same result if we use a different CW structure on $\mathbb{S}^n$ where there are two of each q-cells for all $q \leq n$. The advantage of using this CW structure is that it has a $\mathbb{Z}_2$-action. Further, we can use it to compute the homology of $\mathbb{RP}^n$, see §3.5.

2. Recall that $\mathbb{T}^2$ has a CW structure with one 0-cell, two 1-cells and one 2-cell, see Figure 3.3a. Therefore, $C_*^{cell}(\mathbb{T}^2)$ is

$$\dashrightarrow 0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}\{a, b\} \xrightarrow{d_1} \mathbb{Z}\{p\} \to 0.$$

To compute homology, first, note that $d_1(a) = d_1(b) = 0$. Second, the attaching map

(a) Figure ∞ (b) Geometric realization of $\mathbb{T}^2$

Figure 3.3: CW structures on a torus

$\mathbb{S}^1 \to Sk_1\mathbb{T}^2 \xrightarrow{\cong} Sk_1\mathbb{T}^2/Sk_0\mathbb{T}^2$ sends the boundary of the 2-cell say U to $aba^{-1}b^{-1}$. Therefore, $d_2(U) = a + b - a - b = 0$. With this information, check that we can recover the homology of $\mathbb{T}^2$.

Instead of using the above CW structure, we could have used the geometric realization of $\mathbb{T}^2$ as a semisimplicial set. It consists of one 0-cell, three 1-cells, and two 2-cells, see Figure 3.3b.

Proof of Theorem 3.3.2. For simplicity, we write $Sk_n(X) = X_n$ and drop the superscript of C_*^{cell}. Consider the following diagram where the columns and rows are exact:

$$\begin{tikzcd}
 & C_{n+1}(X) = H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial"] \arrow[dr, "d"] & & H_{n-1}(X_{n-2}) = 0 \arrow[d] \\
0 \arrow[r] & H_n(X_n) \arrow[r, "f"] \arrow[d] & C_n(X) = H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[dr, "d"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
 & H_n(X_{n+1}) \arrow[d] & & C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \\
 & H_n(X_{n+1}, X_n) = 0. & &
\end{tikzcd}$$

First, note that $\partial f = 0$ by exactness which implies $d^2 = 0$. Therefore, $(C_*(X), d)$ is a chain complex.

To compute its homology, note that $\mathrm{Im}(d) = \mathrm{Im}(f\partial) \cong \mathrm{Im}\,\partial$ since f is injective. On the other hand, $\ker(d) = \ker(j\partial) = \ker(\partial) = \mathrm{Im}(f) \cong H_n(X_n)$, where the last first equality is the definition of d (cf. Definition 3.3.1), the second follows because of the injectivity of j, and the last two follow from the exactness of the second row. Therefore,

$$H_n^{cell}(X) = \ker d/\operatorname{Im} d \cong H_n(X_n)/\operatorname{Im}\partial \cong H_n(X_{n+1}) = H_n(X),$$

where the third expression follows from the exactness of the left column in the above diagram while the last one follows from Theorem 3.2.3. □

3.4 Degree of a map of sphere

Recall that we started the discussion of homology to distinguish spaces. In fact, we saw in §2.2 that homology distinguishes maps between spaces. In this section, we will focus on maps between $\mathbb{S}^n$ and the induced maps between homology.

Definition 3.4.1. Fix $1 \leq n \in \mathbb{Z}$. The *degree,* $\deg f$*, of a map* $f : \mathbb{S}^n \to \mathbb{S}^n$ is the image of 1 under $H_n(f) : \mathbb{Z} \to \mathbb{Z}$.

Remark 3.4.2.

1. Degree is multiplicative, i.e., $\deg(g \circ f) = \deg(g)\deg(f)$.
2. If the degree of two maps are different, the maps are not homotopic.
3. We can also define degree for maps on manifolds with same dimension.

Example 3.4.3.

1. The degree of the identity map $\mathbb{1}_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$ is 1.
2. If f is a homeomorphism from $\mathbb{S}^n$ to itself then $\deg(f) = \pm 1$.
3. Check that the map $\mathbb{S}^1 \to \mathbb{S}^1$ that sends $z \mapsto z^d$ has degree d. In some sense, degree of f measures how much it wraps the domain around its image.
4. The degree of a rotation $R_\theta : \mathbb{S}^n \to \mathbb{S}^n$ about an axis by an angle θ is 1 since $R_\theta \simeq \mathbb{1}_{\mathbb{S}^n}$ where the homotopy is obtained by thinking R_θ as a composition of rotations by infinitesimal angles.
5. The degree of the reflection $r : \mathbb{S}^2 \to \mathbb{S}^2$ along the equator is -1. Since the claim is unclear if we use singular homology we use cellular homology. Consider the CW structure on $\mathbb{S}^2$ with two n-cells for $n \leq 2$. Note that r induces a cellular map, $C_*^{cell}(r)$,

$$\begin{tikzcd}
\mathbb{Z}\{A,B\} \arrow[r] \arrow[d, "C_2(r)"] & \mathbb{Z}\{u,v\} \arrow[r] \arrow[d, "C_1(r)"] & \mathbb{Z}\{x,y\} \arrow[d, "C_0(r)"] \\
\mathbb{Z}\{A,B\} \arrow[r] & \mathbb{Z}\{u,v\} \arrow[r] & \mathbb{Z}\{x,y\}
\end{tikzcd}$$

such that $B \mapsto A$ and $A \mapsto B$ which implies $A - B \mapsto B - A$. Further, $C_0(r)$ and $C_1(r)$ are identities. Therefore, $\deg r = -1$.

6. The *antipodal map* $A : \mathbb{S}^2 \to \mathbb{S}^2$ sends $(x, y, z) \mapsto (-x, -y, -z)$. Note that A is a composition of $(x, y, z) \mapsto (-x, y, z) \mapsto (-x, -y, z) \mapsto (-x, -y, -z)$. Since the degree of each map -1, $\deg A = -1$. More generally, we have the following theorem.

Theorem 3.4.4. *The degree of the reflection $r : \mathbb{S}^n \to \mathbb{S}^n$ along the equator is -1 and the degree of antipodal map $A : \mathbb{S}^n \to \mathbb{S}^n$ that maps $z \mapsto -z$ is $(-1)^{n+1}$.*

Proof. Consider the CW structure on $\mathbb{S}^n$ with two i-cells $\{a_i, b_i\}$ for each $i \leq n$. Then $C_*^{cell}(\mathbb{S}^n)$ is

$$0 \to \mathbb{Z}\{a_n, b_n\} \to \mathbb{Z}\{a_{n-1}, b_{n-1}\} \to \cdots \to \mathbb{Z}\{a_0, b_0\} \to 0$$

where $da_n = db_n = b_{n-1} - a_{n-1}$. As with $\mathbb{S}^2$, r induces a cellular map, $C_*(r)$,

$$\begin{tikzcd}
\mathbb{Z}\{a_n, b_n\} \arrow[r] \arrow[d, "C_n(r)"] & \mathbb{Z}\{a_{n-1}, b_{n-1}\} \arrow[r] \arrow[d, "C_{n-1}(r)"] & \dots \arrow[r] & \mathbb{Z}\{a_0, b_0\} \arrow[d, "C_0(r)"] \\
\mathbb{Z}\{a_n, b_n\} \arrow[r] & \mathbb{Z}\{a_{n-1}, b_{n-1}\} \arrow[r] & \dots \arrow[r] & \mathbb{Z}\{a_0, b_0\},
\end{tikzcd}$$

where $C_n(r)(a_n) = b_n$ and $C_n(r)(b_n) = a_n$, otherwise $C_i(r)$ is identity. Therefore, $\deg r = -1$.

To prove that that $\deg A = (-1)^{n+1}$ write $\mathbb{S}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$ and note that A is a composition of reflection along the equator defined by $x_i = 0$. However, if we want to prove from scratch, note that A induces a cellular map, $C_*(A)$,

$$\begin{tikzcd}
\mathbb{Z}\{a_n, b_n\} \arrow[r] \arrow[d, "C_n(A)"] & \mathbb{Z}\{a_{n-1}, b_{n-1}\} \arrow[r] \arrow[d, "C_{n-1}(A)"] & \dots \arrow[r] & \mathbb{Z}\{a_0, b_0\} \arrow[d, "C_0(r)"] \\
\mathbb{Z}\{a_n, b_n\} \arrow[r] & \mathbb{Z}\{a_{n-1}, b_{n-1}\} \arrow[r] & \dots \arrow[r] & \mathbb{Z}\{a_0, b_0\},
\end{tikzcd}$$

where $C_n(A)(b_n) = (-1)^n a_n$ and $C_n(A)(a_n) = (-1)^n b_n$ otherwise $C_i(A)$ is identity. It implies that $\deg A = (-1)^{n+1}$. $\square$

3.5 Real projective space

In this section, we will compute the homology of the real projective space $\mathbb{RP}^n := \mathbb{S}^n / x \sim -x$. Note that $\mathbb{RP}^n$ can be thought of an upper hemisphere of $\mathbb{S}^n$ where the equator is glued via $x \sim -x$, i.e., $\mathbb{RP}^n = \mathbb{D}^n / A_{\partial \mathbb{D}^n}$ where $A_{\partial \mathbb{D}^n}$ is antipodal map on $\partial \mathbb{D}^n$. In particular, $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching the boundary of an n-cell where the attaching map

$\mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$ is the quotient $\sim$, i.e., there is a CW structure on $\mathbb{RP}^n$ with $Sk_k(\mathbb{RP}^n) = \mathbb{D}^k$ for $k \leq n$.

For simplicity, we will compute the homology of $\mathbb{RP}^2$ only. Write c_i as the generator of i-cells. Then the cellular chain complex on $\mathbb{RP}^2$ with boundary maps to be determined is

$$\dashrightarrow 0 \to \mathbb{Z}\{c_2\} \xrightarrow{d_2} \mathbb{Z}\{c_1\} \xrightarrow{d_1} \mathbb{Z}\{c_0\} \to 0. \tag{3.5.1}$$

First, using the CW structure on $\mathbb{S}^2$ with two i-cells $\{a_i, b_i\}$ for each $i \leq 2$, we get a cellular map induced by the quotient $f : \mathbb{S}^2 \to \mathbb{RP}^2$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{a_2, b_2\} \arrow[r] \arrow[d, "C_2(f)"] & \mathbb{Z}\{a_1, b_1\} \arrow[r] \arrow[d, "C_1(f)"] & \mathbb{Z}\{a_0, b_0\} \arrow[r] \arrow[d, "C_0(f)"] & 0 \\
0 \arrow[r] & \mathbb{Z}\{c_2\} \arrow[r] & \mathbb{Z}\{c_1\} \arrow[r] & \mathbb{Z}\{c_0\} \arrow[r] & 0.
\end{tikzcd}$$

If we send $a_i \mapsto c_i$ then we have to send $b_i \mapsto (-1)^i c_i$ since the cellular map has to factor through the diagram of cellular chain complexes

$$\begin{tikzcd}
(\mathbb{Z}\{a_2, b_2\} \to \mathbb{Z}\{a_1, b_1\} \to \mathbb{Z}\{a_0, b_0\}) \arrow[r] \arrow[d, "C_*(f)"] & (\mathbb{Z}\{a_2, b_2\} \to \mathbb{Z}\{a_1, b_1\} \to \mathbb{Z}\{a_0, b_0\}) \arrow[dl, "C_*(g)"] \\
(\mathbb{Z}\{c_2\} \to \mathbb{Z}\{c_1\} \to \mathbb{Z}\{c_0\}) &
\end{tikzcd}$$

induced by the map

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "A"] \arrow[d, "f"] & \mathbb{S}^2 \arrow[dl, "g"] \\
\mathbb{RP}^2, &
\end{tikzcd}$$

where f and g are the quotient maps.

Note that $dc_2 = d(C_2(f)a_2) = C_1(f)(da_2) = C_1(f)(b_1 - a_1) = -2c_1$. On the other hand, $dc_1 = d(C_1(f)a_1) = C_0(f)(b_0 - a_0) = c_0 - c_0$. Therefore, d_2 is multiplication by -2 and $d_1 = 0$ in (3.5.1). Therefore,

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}_2 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.2}$$

In general, we have the following theorem:

Theorem 3.5.1. *For any $n \in \mathbb{N}$, we have*

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k = n \text{ is odd}, \\ \mathbb{Z}_2 & \text{if } 0 < k < n, k \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases} \tag{3.5.3}$$

3.6 Euler characteristic

Going forward, we will study topological invariants other than $H_*(X)$ that can help us answer the Question 1.1.1. In this section, we will study an invariant, *Euler characteristic*, derived from the number of k-cells of CW complexes.

Definition 3.6.1. The *Euler characteristic* χ of a finite CW complex X with k-cells indexed by a set I_k is

$$\chi(X) := \sum_k (-1)^k |I_k|. \tag{3.6.1}$$

Before proving the topological invariance of χ, let's record some group-theoretic language.

Definition 3.6.2. The *rank* $\operatorname{rank} A$ of a finitely generated abelian group A is the number r such that $A \cong \mathbb{Z}^{\oplus r} \bigoplus \oplus_i C_i$ where i runs over a finite set and C_i are cyclic groups.

Remark 3.6.3. Such a decomposition of A always exists and can be proven using elementary group theory.

Lemma 3.6.4. *For any exact sequence $0 \to A \to B \to C \to 0$ in* **Ab**

$$\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C.$$

Theorem 3.6.5. *The Euler characteristic χ of a topological space X is independent of the choice of a (finite) CW structure. In fact,*

$$\chi(X) = \sum_k (-1)^k \operatorname{rank}(H_*(X)). \tag{3.6.2}$$

Proof. Suppose X has a finite CW structure. Let $C_k := C_k^{cell}(X)$, $Z_k := Z_k^{cell}(X)$, and $B_k := B_k^{cell}(X)$. Observe that we have two exact sequences

$$0 \to Z_k \to C_k \to B_{k-1} \to 0,$$
$$0 \to B_k \to Z_k \to H_k \to 0.$$

The first one implies that $\operatorname{rank} C_k = \operatorname{rank} Z_k + \operatorname{rank} B_{k-1}$. Here, $\operatorname{rank} C_k = |I_k|$ which combined with the second exact sequence implies that $\operatorname{rank} C_k = \operatorname{rank}(B_k) + \operatorname{rank}(H_k) + \operatorname{rank}(B_{k-1})$. Then

$$\begin{aligned}\chi(X) &= \sum_k (-1)^k |I_k| \\ &= \sum_k (-1)^k (\operatorname{rank}(B_k) + \operatorname{rank}(H_k) + \operatorname{rank}(B_{k-1})) \\ &= \sum_k (-1)^k \operatorname{rank} H_k.\end{aligned}$$

where we used the fact that each B_k appears twice but with a different sign. □

Example 3.6.6. 1. Note that $\mathbb{S}^2$ has two of each 0-cell, 1-cell and 2-cell. Therefore, $\chi(\mathbb{S}^2) = 2$.

2. Further, $\mathbb{T}^2$ has one 0 cell, two 1-cells and one 2-cell. Therefore, $\chi(\mathbb{T}^2) = 0$. In particular, $\mathbb{T}^2$ and $\mathbb{S}^2$ are not homotopy equivalent.

Chapter 4

Homology with coefficients

Recall the construction of homology groups of a topological space X.

1. First, we defined $\mathbf{Sing}_n(X)$, the space of continuous functions from $\Delta^n \to X$.
2. Then we defined $S_n(X)$ to be the free abelian group $\mathbb{Z}\{\mathbf{Sing}_n(X)\}$ and formed a chain complex $S_*(X)$.
3. Finally, we defined the homology $H(X;\mathbb{Z})$ of X to be the homology of the chain complex $S_*(X)$.

Note that in 2, we could have used a free $\mathcal{R}$-module generated by $\mathbf{Sing}_n(X)$ where $\mathcal{R}$ is a commutative ring (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{F}_p$, the finite field with p elements). In this chapter, we will build a homology theory $H_*(X;\mathcal{R})$ based on any commutative ring $\mathcal{R}$. Later we will see how the choice of $\mathcal{R}$ makes our computation even easier even though they can't distinguish spaces that $H(X;\mathbb{Z})$ can't. In any case, the main goal of this chapter is to answer the following question:

Question 4.0.1. To what extent can we understand $H_q(X;\mathcal{R})$ if we understand $H_q(X;\mathbb{Z})$ or vice versa?

In the process of answering 4.0.1, we will address the following questions, providing an answer to the Question 2.0.1:

Question 4.0.2. How does homology behave under products, i.e., how does $H_q(X \times Y;\mathbb{Z})$ relate to $H_i(X;\mathbb{Z})$ and $H_j(Y;\mathbb{Z})$ where $X, Y \in \mathbf{Top}$?

4.1 Definition of homology revisited

Since we have gone through the grueling work of defining homology $H_*(X;\mathbb{Z})$ with respect to $\mathbb{Z}$, all we have to do is replace $\mathbb{Z}$ with a commutative ring $\mathcal{R}$. In this section, we will compute $H_*(X;\mathcal{R})$ for some space that we have seen so far.

Definition 4.1.1. Let X_* be a semisimplicial set. Then $S_k(X_*;\mathcal{R})$ is the free $\mathcal{R}$-module generated by X_k. Analogously, define a chain complex $S_*(X_*;\mathcal{R})$ of $\mathcal{R}$-modules.

Example 4.1.2. Consider the solid triangle 4.1 with a semisimplicial structure $X_0 = \{a,b,c\}$, $X_1 = \{f,g,h\}$, $X_2 = A$ and $X_i = \emptyset$ for all $i \geq 3$. Then $S_*(X_*;\mathbb{Q})$ is

$$\mathbb{Q}\{A\} \to \mathbb{Q}\{f,g,h\} \to \mathbb{Q}\{a,b,c\}.$$

As before, $\partial f = b - a$, $\partial g = c - b$ and $\partial h = c - a$ and $\partial A = g - h + f$.

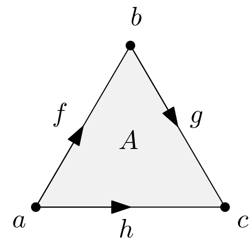


Figure 4.1: Solid triangle

Definition 4.1.3. If X is a topological space, we define $S_*(X;\mathcal{R}) \coloneqq S_*(\mathbf{Sing}_*(X);\mathcal{R})$. Analogously, define $H_*(X;\mathcal{R}) \coloneqq H_*(S_*(\mathbf{Sing}_*(X);\mathcal{R}))$.

Remark 4.1.4.
1. All the theory we have developed so far has $\mathcal{R}$-module analogs with the same proofs. In particular, we can prove the following:
   (a) $H_q(X;\mathcal{R})$ are homeomorphism/homotopy invariants. More generally, all the Eilenberg–Steenrod axioms in §2.7 are true except the dimension axiom. The equivalent of the dimension axiom we get is
   $$H_q(*;\mathcal{R}) \cong \begin{cases} \mathcal{R} & \text{if } q = 0, \\ 0 & \text{else.} \end{cases}$$
   (b) Mayer–Vietoris holds.
   (c) Cellular chain complexes work the same way.
2. The theory of $H(X;\mathcal{R})$ is an example of an extraordinary homology theory.

Example 4.1.5. Recall that $\mathbb{RP}^2 = \mathbb{S}^2/x \sim -x$ has cellular decomposition with one cell for all $i \leq 2$. The corresponding chain complex is $0 \to \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. Analogously, $C_*^{cell}(\mathbb{RP}^2;\mathbb{Q})$ is a chain complex $0 \to \mathbb{Q} \xrightarrow{-2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0$ from which we get

$$H_i(\mathbb{RP}^2;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $C_*^{cell}(\mathbb{RP}^2;\mathbb{F}_2)$ is a chain complex $0 \to \mathbb{F}_2 \xrightarrow{-2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to 0$ from which we get

$$H_i(\mathbb{RP}^2;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } i \in \{0,1,2\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.1}$$

Remark 4.1.6. 1. In contrast to $H_1(\mathbb{RP}^2;\mathbb{Z})$, $H_1(\mathbb{RP}^2;\mathbb{Q})$ does not detect 1-cell while $H_i(\mathbb{RP}^2;\mathbb{F}_2)$ detects the 2-cell.

2. Note that ± 1 are the same element in $\mathbb{F}_2$. Therefore, we can resolve the sign problem we stated in Remark 2.6.5 using $H_*(X;\mathbb{F}_2)$.

3. In a real-life application, one starts with a data set and gives it a semisimiplicial structure X_ϵ depending on a parameter ϵ. Precisely, for fixed $\epsilon > 0$, one connects two points in the data set by an edge if they are ϵ-close and connects three points by a triangle if they lie in an ϵ-ball. In general, if $n+1$ points are ϵ-close then one forms an n-simplex. One question to study is the dependence of $H_q(X_\epsilon;\mathcal{R})$ on the parameter ϵ (look for persistent homology).

4. In complexity theory, one tries to understand the time it takes to compute $H_q(X;\mathcal{R})$ for different $\mathcal{R}$. In fact, it is known that $H_q(X;\mathbb{Q})$ is faster to compute than $H_q(X;\mathbb{Z})$.

5. Define the *configuration space* $\mathrm{Config}_k(\mathbb{T}^2)$ corresponding to a system of points moving on $\mathbb{T}^2$ as

$$\mathrm{Config}_k(\mathbb{T}^2) \coloneqq \{(x_1, x_2 \dots x_k) \in (\mathbb{T}^2)^k \mid x_i \neq x_j \text{ for } i \neq j\}/\mathbf{S}_k.$$

where $\mathbf{S}_k$ is the symmetric group of order k. In physics, one asks how $H_q(\mathrm{Config}_k(\mathbb{T}^2);\mathcal{R})$ behaves as a function of q and k. As of now, the homology groups are known when $\mathcal{R} = \mathbb{Q}$ or $\mathbb{F}_2$ but the case $\mathcal{R} = \mathbb{F}_3$ is still wide open.

4.2 Tensor products

In this section, we will set up a categorical language for (tensor) products (cf. Theorem 4.2.5) so that we can start answering the Question 4.0.1, especially to phrase products of topological spaces at the level of chain complexes.

There is a category $\mathcal{R}$-**mod** of $\mathcal{R}$-modules with $\mathcal{R}$-linear maps as morphisms. Note that for any $A, B \in \mathrm{ob}(\mathcal{R}\text{-}\mathbf{mod})$, the space of morphisms $\underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(A,B)$ has a structure of $\mathcal{R}$-module. Namely, if $f, g : A \to B$ are maps of $\mathcal{R}$-modules, then $f + g : A \to B$ is also a map of $\mathcal{R}$-modules. Furthermore, if $r \in \mathcal{R}$ then $rf : A \to B$ is a map of $\mathcal{R}$-modules where $(rf)(a) \coloneqq rf(a)$ for any $a \in A$.

Note the composition $A \to B_1 \to B_2$ of maps of $\mathcal{R}$-modules is also a map of $\mathcal{R}$-modules. In particular, for any $A \in \mathcal{R}\text{-}\mathbf{mod}$, we can view $\underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(A,\bullet) : \mathcal{R}\text{-}\mathbf{mod} \to \mathcal{R}\text{-}\mathbf{mod}$ as a pullback functor. Similarly, the composition $A_1 \to A_2 \to B$ $\mathcal{R}$-modules induces a push forward functor $\underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(\bullet, B) : (\mathcal{R}\text{-}\mathbf{mod})^{op} \to \mathcal{R}\text{-}\mathbf{mod}$. To summarize, there is a functor $\underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(\bullet,\bullet) : (\mathcal{R}\text{-}\mathbf{mod})^{op} \times (\mathcal{R}\text{-}\mathbf{mod}) \to \mathcal{R}\text{-}\mathbf{mod}$.

In general, any category $\mathcal{C}$ induces a functor $\mathbf{Hom}_{\mathcal{C}}(\bullet,\bullet) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$. Usually, when there is a structure of $\mathcal{C}$ on $\mathbf{Hom}_{\mathcal{C}}(\bullet,\bullet)$ the functor bears a name.

Definition 4.2.1. An *internal* **Hom** *on a category* $\mathcal{C}$ is a functor $\underline{\mathbf{Hom}}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$.

Example 4.2.2. There **Hom** functors are internal **Hom**'s in $\mathcal{R}$-**mod** and **Set**.

Remark 4.2.3. An internal **Hom** does not always have to be the same as $\mathbf{Hom}_{\mathcal{C}}$. In fact, the internal **Hom** in $\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})$ is different than $\mathbf{Hom}_{\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})}$ (see (4.4.2)). To be aware of this distinction, in literature, one often writes $[\bullet, \bullet]$ to denote an internal **Hom**. However, we will just write $\underline{\mathbf{Hom}}_{\mathcal{C}}$ where the underline emphasizes that we are talking about an internal **Hom**.

In the rest of this section, we will see that the algebra of internal **Hom** is tied to homology groups with coefficients. Further, the category with an internal **Hom** allows us to naturally define tensor products.

Remark 4.2.4. There is a *currying isomorphism*:

$$\mathbf{Hom}_{\mathbf{Set}}(A \times B, C) \cong \mathbf{Hom}_{\mathbf{Set}}(A, \mathbf{Hom}_{\mathbf{Set}}(B, C)).$$

That is, given $f : A \times B \to C$ we can define a morphsim $g : A \to \mathbf{Hom}(B, C)$ by setting $(g(a))(b) \coloneqq f(a, b)$. Check that the isomorphism is natural in A, B and C.

Theorem 4.2.5 (Tensor products). *Suppose $\mathcal{R}$ is a commutative ring. There exists a functor*

$$\otimes_{\mathcal{R}} : \mathcal{R}\text{-}\mathbf{mod} \times \mathcal{R}\text{-}\mathbf{mod} \to \mathcal{R}\text{-}\mathbf{mod}$$

with $\otimes_{\mathcal{R}}(M, N)$ written as $M \otimes_{\mathcal{R}} N$ such that there is a natural isomorphism

$$\underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(M \otimes_{\mathcal{R}} N, P) \cong \underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(M, \underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(N, P))$$

*for every $P \in \mathcal{R}$-**mod**. Further, the existence of such a natural isomorphism uniquely specifies $\otimes_{\mathcal{R}}$.*

Before proving Theorem 4.2.5, let's first recall the definition of tensor products used in algebra.

Definition 4.2.6. The *tensor product* of $M, N \in \mathcal{R}$-**mod** written as $M \otimes_{\mathcal{R}} N$ is the free $\mathcal{R}$-module generated by symbols $m \otimes n$ where $m \in M$ and $n \in N$ modulo the following relations.

- **Distributivity:** $m \otimes (n + n') = m \otimes n + m \otimes n'$ and $(m + m') \otimes n = m \otimes n + m' \otimes n$.
- **Scalar multiplication:** $rm \otimes n = (rm) \otimes n = m \otimes (rn)$.

Remark 4.2.7. Theorem 4.2.5 means that tensor product is the same as the internal **Hom** $\otimes_{\mathcal{R}}$ reinterpreted using the currying isomorphism. Precisely, $\otimes_{\mathcal{R}}$ is adjoint to the internal **Hom** as functor.

Sketch of the proof of Theorem 4.2.5. Given $f : M \otimes_{\mathcal{R}} N \to P$, we define $g : M \to \underline{\mathbf{Hom}}_{\mathcal{R}}(N, P)$ by $(g(m))(n) = f(m \otimes n)$. Check that this map is a bijection. □

Example 4.2.8. As an example of the existence of a natural isomorphism in Theorem 4.2.5 let's prove

$$\underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbf{Hom}_{\mathcal{R}\text{-}\mathbf{mod}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})). \tag{4.2.1}$$

Note that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ has generators $0\otimes 0, 0\otimes 1, 0\otimes 2, 0\otimes 3, 1\otimes 0, 1\otimes 1, 1\otimes 2, 1\otimes 3$. Note that $0 \otimes i = 0$. Therefore, the first five elements are $0 \otimes 0$. On the other hand $1 \otimes 3 = 1 \otimes 1 + 1 \otimes 2$. Hence, the group is generated by $1 \otimes 1$. That is, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. Therefore,

$$\underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}. \tag{4.2.2}$$

On the other hand, $\underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore,

$$\underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/4\mathbb{Z}, \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Remark 4.2.9.

1. Although the symmetry $M \otimes_{\mathcal{R}} N \cong N \otimes_{\mathcal{R}} M$ where $m \otimes n \mapsto n \otimes m$ easily follows from the definition of tensor products, it is unclear from its interpretation as the internal **Hom** in Theorem 4.2.5.

2. However, there is an advantage of the interpretation of tensor products as internal **Hom** because it allows us to understand $A \otimes_{\mathbb{Z}} \mathbb{Z}$ for arbitrary $A \in \mathbf{Ab}$. Note that when we don't have explicit information about the generators of A we can use Definition 4.2.6 to compute $A \otimes_{\mathbb{Z}} \mathbb{Z}$. In any case, for any $B \in \mathbf{Ab}$, we have

$$\underline{\mathbf{Hom}}_{\mathbf{Ab}}(A \otimes_{\mathbb{Z}} \mathbb{Z}, B) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(A, \underline{\mathbf{Hom}}_{\mathbf{Ab}}(\mathbb{Z}, B)) \cong \underline{\mathbf{Hom}}_{\mathbf{Ab}}(A, B).$$

   Therefore, the following Lemma implies that $A \otimes_{\mathbb{Z}} \mathbb{Z} \cong A$.

Lemma 4.2.10 (Yoneda). *For all $X, Y \in \mathbf{Ab}$, if $\mathbf{Hom}(X, B) \cong \mathbf{Hom}(Y, B)$ via an isomorphism natural in B then $X \cong Y$.*

Proof. Left as an exercise. □

Example 4.2.11. Using either generators or Theorem 4.2.5 we can prove that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$. More generally, we have the following proposition.

Proposition 4.2.12. *Denote $\gcd(m, n)$ to be the greatest common divisor of $m, n \in \mathbb{Z}$. Then*

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}.$$

Proposition 4.2.13. *For any commutative ring $\mathcal{R}$ and $M, N, P \in \mathcal{R}$-**mod***

$$M \otimes_{\mathcal{R}} (N \oplus P) \cong (M \otimes_{\mathcal{R}} N) \oplus (M \otimes_{\mathcal{R}} P),$$
$$(M \oplus N) \otimes_{\mathcal{R}} P \cong M \otimes_{\mathcal{R}} P \oplus N \otimes_{\mathcal{R}} P.$$

Remark 4.2.14. 1. Proposition 4.2.13 implies that $\mathcal{R}$**-mod** has a categorical ring structure with $\oplus$ as addition and $\otimes_{\mathcal{R}}$ as multiplication.

2. Every finitely generated abelian group can be written as copies of $\mathbb{Z}$ and cyclic groups. Therefore, using Proposition 4.2.13, we can understand the tensor products of any finitely generated abelian groups.

Example 4.2.15. Using the distribution law, we see that $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proposition 4.2.16 (Right exactness of $\otimes$). *Suppose A, B, C and M are $\mathcal{R}$-modules. If $A \to B \to C \to 0$ is exact then $A \otimes M \to B \otimes M \to C \otimes M \to 0$ is exact for any M.*

Proof. Left as an exercise. □

Remark 4.2.17. In general, we don't have the left exactness for $\otimes$. In fact, consider $0 \to \mathbb{Z}\overset{\times t}{\mathbb{Z}}$. Let $M = \mathbb{Z}/2\mathbb{Z}$. Then $\mathbb{Z} \otimes M \overset{\times 2 \otimes \mathbb{1}}{\to} \mathbb{Z} \otimes M$ is a zero map. However, if all of the modules are free then we get left-exactness.

Proposition 4.2.18 (Exactness of Hom). *Suppose A, B, C and M are free $\mathcal{R}$-modules.*

1. *If $0 \to A \to B \to C$ is exact then $0 \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(M, A) \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(M, B) \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(M, C)$ is exact for any M.*

2. *$A \to B \to C \to 0$ is exact if and only if $0 \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(C, M) \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(B, M) \to \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(A, M)$ is exact for any M.*

Proof. We leave the first statement as an exercise. To prove the second one, suppose $f \in \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(B, M)$ goes to 0 in $\underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(A, M)$ meaning $f \circ p = 0$ where $p : A \to B$. Since $C = B/\operatorname{Im} A$, f factors through C, i.e., $f = gq$ for some $g \in \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(C, M)$. This gives proof of exactness in the forward direction. The other direction is left as an exercise. □

In the rest of this section, we will generalize the definition of tensor products to chain complexes which will allow us to understand the homology groups of more product spaces.

Definition 4.2.19. The *tensor product* $C_* \otimes_{\mathcal{R}} D_*$ of chain complexes C_*, D_* of $\mathcal{R}$-modules is the chain complex with

$$(C_* \otimes_{\mathcal{R}} D_*)_n := \bigoplus_{p+q=n} C_p \otimes_{\mathcal{R}} D_q$$

such that for any $c_p \in C_p$ and $d_q \in D_q$ the boundary maps ∂ are defined as

$$\partial(c_p \otimes d_q) := (\partial c_p) \otimes d_q + (-1)^p c_p \otimes (\partial d_q). \tag{4.2.3}$$

Remark 4.2.20. The sign $(-1)^p$ appears (4.2.3) because it makes ∂ well defined. Geometrically, it has to be consistent with the fact that the tensor product of cellular chain complexes associated with finite chain complexes X and Y is the same as the product chain complex associated with $X \times Y$. In fact, $X \times Y$ has a CW structure with one $p+q$-cell for each pair formed by p-cell of X and q-cell of Y. The boundary map d is defined by $d \cong d^p \times d^q \to X \times Y$. Then,

$$C_*^{cell}(X \times Y) \cong C_*^{cell}(X) \otimes C_*^{cell}(Y). \tag{4.2.4}$$

Example 4.2.21.

1. Consider $I_0 \times I_1$ where I_i is an interval $e_i \xrightarrow{a_i} f_i$. Note that $C_*^{cell}(I_0 \times I_1) = C_*^{cell}(I_0) \otimes_{\mathbb{Z}} C_*^{cell}(I_1)$, see Fig 4.2. Here,

$$\begin{aligned}\partial(a_0 \otimes a_1) &= (\partial a_0) \otimes a_1 - a_0 \otimes (\partial a_1)\\ &= f_0 \otimes a_1 - e_0 \otimes a_1 - a_0 \otimes f_1 + a_0 \otimes e_1.\end{aligned}$$

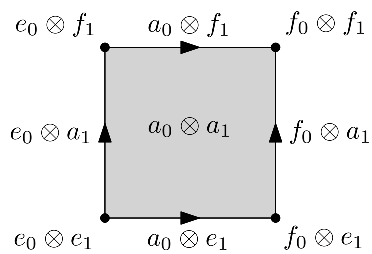


Figure 4.2: $I_0 \otimes I_2$

2. Fix $X \in$ **Top**. Define $C_*(X) = S_*(X)$ and D_* such that $D_0 = A$ and $D_n = 0$ for all $n \neq 1$. Then $C_* \otimes D_*$ is the chain complex

$$\dashrightarrow S_n(X) \otimes A \to S_{n-1}(X) \otimes A \dashrightarrow .$$

Definition 4.2.22. There is an internal **Hom** on **ch**($\mathcal{R}$-**mod**) such that for any $X_*, Y_* \in$ **ch**($\mathcal{R}$-**mod**) with corresponding boundary maps ∂_X, ∂_Y we have

$$(\underline{\mathbf{Hom}}_{\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})}(X,Y))_n := \bigoplus_{i \in \mathbb{Z}} \mathbf{Hom}(X_i, Y_{i+n}) \tag{4.2.5}$$

and the boundary maps are defined such that for $f \in (\underline{\mathbf{Hom}}_{\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})}(X,Y))_n$

$$\partial f := \partial_Y \circ f - (-1)^n f \circ \partial_X. \tag{4.2.6}$$

Remark 4.2.23.

1. There is a technical subtlety in (4.2.5) where we have to replace $\bigoplus$ by $\prod$ in general. However, the concepts agree when the complexes involved Z_* are *bounded*, i.e., there is some $m \in \mathbb{Z}$ such that $Z_n = 0$ for all $n \geq m$ or for all $n \leq m$. Since we will mostly deal with *bounded* chain complexes we will ignore the subtlety going forward.

2. Check the analog of Theorem 4.2.5 for $\underline{\mathbf{Hom}}_{\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})}$ and $\otimes_{\mathcal{R}}$, i.e., the internal **Hom** and $\otimes_{\mathcal{R}}$ are the same up to a re-interpretation (they are adjoint functors).

4.3 Fundamental theorem of homological algebra

In the next couple of sections, we will record some technical details from homological algebra before we provide an answer (cf. Theorem 4.5.7 and Theorem 4.6.1) to the questions posed below.

Note that (4.2.4) implies that the homology of $X \times Y$ can be determined using the homology of X and Y. In general, we can ask the following question.

Question 4.3.1. Given two chain complexes C_*, D_*, can we express $H_*(C_* \otimes D_*)$ in terms of $H_*(C_*)$ and $H_*(D_*)$?

We also have a topological version of the Question 4.3.1:

Question 4.3.2. Can we compute $H_n(X \times Y)$ by computing the homology of $S_n(X) \otimes S_n(Y)$?

In general, the answer to the Question 4.3.1 is no. For instance, suppose C_* is the chain complex $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to 0$, C'_* is the chain complex $0 \to 0 \to \mathbb{Z}/p\mathbb{Z} \to 0$ and D_* is the chain complex $0 \to \mathbb{Z}/p\mathbb{Z} \to 0 \to 0$. Then there exists a chain map $C_* \to C'_*$ that induces homology isomorphism, but the induced map $C_* \otimes D_* \to C'_* \otimes D_*$ is not a homology isomorphism. However, we can canonically replace a module by its *free resolution* (Theorem 4.3.8). Then the idea is to use free resolutions to relate the homology of $C_* \otimes D_*$ in terms of the homology of C_* and D_* of (cf. Theorem 4.5.3 and Theorem 4.5.7).

Definition 4.3.3. A chain map $f : C_* \to D_*$ is a *quasi-isomorphism* if it induces a homology isomorphism $H_q(f) : H_q(C_*) \to H_q(D_*)$ for all q.

Example 4.3.4. The following map is a quasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 5"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Definition 4.3.5. A *free resolution of an $\mathcal{R}$-module M* is a chain complex C_* of free $\mathcal{R}$-module such that there is a quasi-isomorphism $C_* \to M$, where M is viewed as a chain complex concentrated at 0^{th} degree.

Remark 4.3.6.
1. If $\mathcal{R}$ is a principal ideal domain (PID) then every $\mathcal{R}$-module M admits a free resolution $0 \to C_1 \hookrightarrow C_0 \to 0$. Define C_0 to be the free $\mathcal{R}$-module generated by the generators of M; unless M is finitely generated one has to assume the axiom of choice to find a generator of M. In any case, define $\epsilon_M : C_0 \to M$ by sending the generators of C_0 to the generators of M. Set $C_1 = \ker \epsilon_M$. Note that C_1 is free since a submodule of a free module is always free given that $\mathcal{R}$-is a PID (see [Rot10]).

2. If $\mathcal{R}$ is not a PID, we can still find a free resolution, not necessarily a finite one.

3. However, a free resolution is not unique, see Example 4.3.7 2.

Example 4.3.7. 1. If $\mathcal{R}$ is a field, every $\mathcal{R}$-module is a free resolution of itself.

2. Suppose $\mathcal{R} = \mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The following diagram such that $f((0,1)) = (0,0,2)$ and $f((1,0)) = (3,0,0)$ is a quasi-isomorphism:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Another free resolution of M is the following diagram such that $g_1(1) = (0,0,1)$ and $g_2((1,0,0)) = (3,0,0)$, $g_2((0,1,0)) = (0,0,2)$ and $g_2((0,0,1)) = (0,0,0)$:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "g_1"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "g_2"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

3. Suppose $\mathcal{R} = \mathbb{Q}[t]/t^2$ and $M = \mathbb{Q}$ where t acts by 0. A free resolution of M exists but with infinitely many non-zero degree.

Theorem 4.3.8 (Fundamental theorem of homological algebra). *Suppose M and N are $\mathcal{R}$-modules with free resolutions given by the following commutative diagrams*

$$\begin{tikzcd}
{} \arrow[r, dotted] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
{} \arrow[r, dotted] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
{} \arrow[r, dotted] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
{} \arrow[r, dotted] & 0 \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Then any $\mathcal{R}$-module map $f : M \to N$ can be lifted to a chain map $f_* : F_* \to E_*$. Furthermore, f_* is unique up to chain homotopy.

Proof. Suppose $f : M \to N$ is a map of $\mathcal{R}$-modules. First, we have to find a map $f_0 : F_0 \to E_0$ so that the following diagram commutes.

$$\begin{tikzcd}
F_0 \arrow[r, two heads, "\epsilon_M"] \arrow[d, dashed, "f_0"] & M \arrow[d, "f"] \\
E_0 \arrow[r, two heads, "\epsilon_N"] & N
\end{tikzcd}$$

Since F_0 and E_0 are free, it suffices to find a map for generators of F_0. Note that ϵ_M and ϵ_N are surjective since $H_0(F_*) = F_0/(\operatorname{Im} F_1 \to F_0) \cong M$ and $H_0(E_*) = E_0/(\operatorname{Im} E_1 \to E_0) \cong N$. Therefore, we can send a generator g_0 of F_0 to an arbitrary lift via ϵ_N of $f(\epsilon_M(g_0))$.

On the other hand, the free resolution diagram implies that F_1 factors through $\ker \epsilon_M$ and E_1 through $\ker \epsilon_N$, see the diagram below. Further, since $F_0/\ker \epsilon_M \cong M$ and $E_0/\ker \epsilon_N \cong N$ we know that $F_1 \twoheadrightarrow \ker \epsilon_M$ and $E_1 \twoheadrightarrow \ker \epsilon_N$ are surjective maps. Now we can define f_1 using the following diagram:

$$
\begin{tikzcd}
F_1 \arrow[r, two heads, "\epsilon_M"] \arrow[d, dashed, "f_1"] & \ker \epsilon_M \arrow[d, "f_0"] \\
E_1 \arrow[r, two heads, "\epsilon_N"] & \ker \epsilon_N
\end{tikzcd}
$$

In general, the free resolution diagram implies that $F_n \twoheadrightarrow \ker(F_{n-1} \to F_{n-2})$ and $E_n \twoheadrightarrow \ker(E_{n-1} \to E_{n-2})$ are surjective maps. Therefore, we can inductively define f_n using the following diagram

$$
\begin{tikzcd}
F_n \arrow[r, two heads, "\epsilon_M"] \arrow[d, dashed, "f_n"] & \ker(F_{n-1} \to F_{n-2}) \arrow[d, "f_{n-1}"] \\
E_n \arrow[r, two heads, "\epsilon_N"] & \ker(E_{n-1} \to E_{n-2})
\end{tikzcd}
$$

By construction $f_* : F_* \to E_*$ is a chain map.

$$
\begin{tikzcd}
 & & F_0 \arrow[r, "\epsilon_M"] \arrow[d, "g_0"'] \arrow[dl] \arrow[dll, dotted, "h"'] & M \arrow[d, "0"] \\
E_1 \arrow[r, two heads] & \ker \epsilon_N \arrow[r] & E_0 \arrow[r, "\epsilon_N"] & N
\end{tikzcd}
$$

Now suppose that $f_*, f'_* : F_* \to E_*$ are two lifts of $f : M \to N$. We have to construct a chain homotopy h such that $\partial h + h\partial = f_* - f'_* =: g_*$ where g_* is the lift of 0. Note that $\epsilon_N g_0 = 0$. Therefore, g_0 factors through $\ker \epsilon_N$. Now define $h : F_0 \to E_1$ to be a lift of g_0. By construction, $\partial h = g_0$.

$$
\begin{tikzcd}
 & & F_1 \arrow[r, "\partial"] \arrow[d, "g_1"'] \arrow[dl] \arrow[dll, dotted, "h"'] & F_0 \arrow[d, "g_0"] \arrow[dl, "h"'] \\
E_2 \arrow[r, two heads] & \ker(E_1 \to E_0) \arrow[r] & E_1 \arrow[r, "\partial"'] & E_0
\end{tikzcd}
$$

Further, note that

$$\partial(g_1 - h\partial) = \partial g_1 - \partial h\partial = g_0\partial - \partial h\partial = (g_0 - \partial h)\partial = 0$$

Therefore, $g_1 - h\partial$ factors through $\ker(E_1 \to E_0)$, see the diagram above. Now define $h : F_1 \to E_2$ by lifting $g_1 - h\partial$ via the surjection $E_2 \twoheadrightarrow \ker(E_1 \to E_0)$. By construction $\partial h = g_1 - h\partial$.

In general, we have

$$\partial(g_n - h\partial) = (g_{n-1} - \partial h)\partial = h\partial\partial = 0.$$

Therefore, $g_n - h\partial$ factors through $\ker(E_n \to E_{n-1})$, see the diagram below. Now define $h : F_n \to E_{n+1}$ using the lift of $g_n - h\partial$. Therefore, h is the desired chain homotopy between f_* and f'_*.

$$\begin{tikzcd}
 & & F_n \arrow[r, "\partial"] \arrow[dll, dotted, "h"'] \arrow[dl] \arrow[d, "g_n"'] & F_{n-1} \arrow[dl, "h"'] \arrow[d, "g_{n-1}"] \\
E_{n+1} \arrow[r, two heads] & \ker(E_n \to E_{n-1}) \arrow[r] & E_n \arrow[r, "\partial"'] & E_{n-1}.
\end{tikzcd}$$

□

4.4 Derived category

Recall that the homology functors on **Top** factor through the homotopy category **Ho**(**Top**) (cf. Theorem 2.2.8). A similar algebraic situation we have is quasi-isomorphic chain complexes have the same homology groups. The algebraic analog of **Ho**(**Top**) is the *derived category* $\mathcal{D}(\mathcal{R})$, i.e., homology functors on the category **ch**($\mathcal{R}$-**mod**) of chain complexes of $\mathcal{R}$-**mod** factors through $\mathcal{D}(\mathcal{R})$. In this section, we will scratch the surface of derived category and functors glossing over some set-theoretic issues. For a detailed exposition see [Wei95]. In any case, our goal is to develop a language so that we can refine the Question 4.3.1 to the following question:

Question 4.4.1. Is $S_*(X \times Y; \mathcal{R})$ isomorphic to $S_*(X;\mathcal{R}) \otimes_{\mathcal{R}} S_*(Y;\mathcal{R})$ in $\mathcal{D}(\mathcal{R})$?

Definition 4.4.2. The "*derived category*" $\mathcal{D}(\mathcal{R})$ of **ch**($\mathcal{R}$-**mod**) consists of

- ob($\mathcal{D}(\mathcal{R})$) = ob(**ch**($\mathcal{R}$-**mod**)), and
- in addition to the morphisms in **ch**($\mathcal{R}$-**mod**), we include formal inverses of quasi-isomorphism in **ch**($\mathcal{R}$-**mod**).

Remark 4.4.3.
1. We can think of the construction of derived category as building integers from natural numbers by adding formal additive inverses (negatives) or building fractions from integers by adding multiplicative inverses.
2. There are many morphisms in $\mathcal{D}(\mathcal{R})$ that are not in $\mathbf{ch}(\mathcal{R}\textbf{-mod})$.
3. An actual construction of the derived category is out of the scope of these notes because we will run into set-theoretic issues, see [Wei95]. However, we won't need the construction to see the consequences of $\mathcal{D}(\mathcal{R})$.

Similar to $\mathbf{ch}(\mathcal{R}\textbf{-mod})$, the derived category $\mathcal{D}(\mathcal{R})$ has a tensor product denoted by $\otimes^{\mathbb{L}}_{\mathcal{R}}$. Here, $\mathbb{L}$ emphasizes the fact that the tensor product appears on the left side of the currying isomorphism.

Definition 4.4.4. The *tensor product* $C_* \otimes^{\mathbb{L}}_{\mathcal{R}} D_*$ of $C_*, D_* \in \mathcal{D}(\mathcal{R})$ is

- $C_* \otimes_{\mathcal{R}} D_*$ if either C_* or D_* are chain complexes of free $\mathcal{R}$-modules, and
- in general, we replace C_* or D_* by quasi-isomorphic chain complexes of free $\mathcal{R}$-modules.

Remark 4.4.5.
1. The extension of a functor F on $\mathbf{ch}(\mathcal{R}\textbf{-mod})$ to $\mathcal{D}(\mathcal{R})$ is called the *derived functor* of F.[1] In particular, $\otimes^{\mathbb{L}}_{\mathcal{R}}$ is the derived functor of $\otimes_{\mathcal{R}}$.
2. Analogous to Theorem 4.2.5, there is a characterization of $\otimes^{\mathbb{L}}_{\mathcal{R}}$ as the adjoint of an internal **Hom** on $\mathcal{D}(\mathcal{R})$.

Definition 4.4.6. Suppose $M, N \in \mathcal{R}\textbf{-mod}$. By considering M, N to be chain complexes concentrated at deg 0, we define

$$\mathrm{Tor}_i(M, N) \coloneqq H_i(M \otimes^{\mathbb{L}}_{\mathcal{R}} N). \tag{4.4.1}$$

Remark 4.4.7.
1. We could have defined $\mathrm{Tor}_i(M, N)$ simply as $H_i(F \otimes_{\mathcal{R}} N)$ where F is a free resolution of M, without appealing to the derived category. This definition is well-defined, see Theorem 4.4.9, and that's the approach taken in [Hat05] and [Mil16]. However, the language of the derived category and derived functor make Tor more conceptual.
2. We can and will define Tor in the next chapter (cf. Remark 5.1.4) using the internal **Hom** in $\mathcal{D}(\mathcal{R})$[2] that descends from the internal **Hom** in $\mathbf{ch}(\mathcal{R}\textbf{-mod})$, see Definition 4.2.22. Note that when F_* is a chain complex of free $\mathcal{R}$-modules and $N \in \mathcal{R}\textbf{-mod}$ then we have

$$(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathcal{R})}(F_*, N))_i = \underline{\mathbf{Hom}}_{\mathcal{R}\textbf{-mod}}(F_{-i}, N). \tag{4.4.2}$$

The sign appears because of the contravariance of **Hom** .

[1] In literature, Tor is called the derived functor of $\otimes_{\mathcal{R}}$, the tensor product in $\mathbf{ch}(\mathcal{R}\textbf{-mod})$. However, we follow the convention of [Wei95] where, in the setting of derived category, the author writes hyper-derived functors to mean what in literature is often called derived functors. In particular, Tor is a hyper-derived functor of $\otimes_{\mathcal{R}}$.

[2] $\underline{\mathbf{Hom}}_{\mathcal{D}(\mathcal{R})}$ is the derived functor of $\underline{\mathbf{Hom}}_{\mathbf{ch}(\mathcal{R}\textbf{-mod})}$.

Example 4.4.8. 1. If $\mathcal{R}$ is a field $\mathrm{Tor}_i(M,N) = 0$ for all $i > 0$ where M and N are $\mathcal{R}$-modules.

2. If $\mathcal{R} = \mathbb{Z}$ and M, N are finitely generated $\mathbb{Z}$-modules then $\mathrm{Tor}_i(M,N) = 0$ for all $i > 1$.[3]

3. If $M \in \mathcal{R}$-**mod** then $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} M = M$. Therefore, $\mathrm{Tor}_0(\mathbb{Z}, M) = M$ and $\mathrm{Tor}_i(\mathbb{Z}, M) = 0$ for all $i \geq 1$.

4. We will compute $\mathrm{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ using two different resolutions of $\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$.

   - **Method 1:** Note that $\mathbb{Z}/2\mathbb{Z}$ has a free resolution $F_* = (0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0)$. Check that the tensor product $F_* \otimes \mathbb{Z}/4\mathbb{Z}$ is given by $0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \to 0$. Thus, $\mathrm{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for $i \in \{0, 1\}$ and 0 otherwise.
   - **Method 2:** Taking the free resolutions of both terms we get that $\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ is $(\mathbb{Z}\{a\} \xrightarrow{2} \mathbb{Z}\{b\} \to 0) \otimes (\mathbb{Z}\{c\} \xrightarrow{4} \mathbb{Z}\{d\} \to 0)$ which is equal to $\mathbb{Z}\{a \otimes c\} \to \mathbb{Z}\{b \otimes c, a \otimes d\} \to \mathbb{Z}\{b \otimes d\} \to 0$. Compute the boundary maps and check that Tor_i agrees with the computation done before.

5. To compute $\mathrm{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$, let's use the free resolution of $\mathbb{Z}/2\mathbb{Z}$, namely $(\mathbb{Z} \xrightarrow{2} \mathbb{Z})$ Then $\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is $0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{2} \mathbb{Z}/3\mathbb{Z} \to 0$. Check that $\mathrm{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0$. In general, we get

$$\mathrm{Tor}_i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\gcd(m,n)\mathbb{Z} \tag{4.4.3}$$

   when $i \in \{0, 1\}$ and 0 else.

The following theorem states the independence of $\mathrm{Tor}_i(M,N)$ on the free resolution of M (and similarly of N). Therefore, (4.4.1) is well defined.

Theorem 4.4.9. *Suppose $M, N \in \mathcal{R}$-**mod**. Assume that $F_* \xrightarrow{\epsilon} M$ and $F'_* \xrightarrow{\epsilon'} M$ are two free resolutions of M. Then* $\mathrm{Tor}_i(M,N) = H_i(F_* \otimes_{\mathcal{R}} N) \cong H_*(F'_* \otimes_{\mathcal{R}} N)$.

Proof. First, choose chain map $f : F_* \to F'_*$ that lifts $\mathbb{1}_M$. Similarly, suppose $g : F'_* \to F_*$ lifts $\mathbb{1}_M$. Note that $(g \circ f) : F_* \to F_*$ lifts $\mathbb{1}_M$, so by Theorem 4.3.8 it is chain homotopic to $\mathbb{1}_{F_*}$. Similarly, $f \circ g$ is chain homotopic to $\mathbb{1}_{F'_*}$. In particular, we get two maps $f \otimes_{\mathcal{R}} \mathbb{1}_N : F_* \otimes_{\mathcal{R}} N \to F'_* \otimes_{\mathcal{R}} N$ and $g \otimes_{\mathcal{R}} \mathbb{1}_N : F'_* \otimes_{\mathcal{R}} N \to F_* \otimes_{\mathcal{R}} N$. Note that $(g \otimes_{\mathcal{R}} \mathbb{1}_N) \circ (f \otimes_{\mathcal{R}} \mathbb{1}_N) = (g \circ f) \otimes_{\mathcal{R}} \mathbb{1}_N$ is chain homotopic to $\mathbb{1}_{F_* \otimes_{\mathcal{R}} N}$. Similarly $(f \otimes_{\mathcal{R}} \mathbb{1}_N) \circ (g \otimes_{\mathcal{R}} \mathbb{1}_N)$ is chain homotopic to $\mathbb{1}_{F'_* \otimes_{\mathcal{R}} N}$. Thus, $f \otimes_{\mathcal{R}} N$ and $g \otimes_{\mathcal{R}} N$ are inverse quasi-isomorphism which proves the theorem. $\square$

[3] In the context of commutative algebra, Tor reflects the fact that a field is 0-dimensional and $\mathbb{Z}$ is one-dimensional.

4.5 Universal coefficient theorem

In this section, we will address the Question 4.0.1. Further, we will write $\mathrm{Tor}^{\mathcal{R}}$ to emphasize that we are working on $\mathcal{D}(\mathcal{R})$.

Definition 4.5.1. For any topological space X and a $\mathbb{Z}$-module (abelian group) M, we define

$$\begin{aligned} S_*(X;M) &\coloneqq S_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} M \cong S_*(X;\mathbb{Z}) \otimes^{\mathbb{L}}_{\mathbb{Z}} M, \\ H_*(X;M) &\coloneqq H_*(S_*(X;M)). \end{aligned}$$

Remark 4.5.2. If M is the underlying abelian group of a ring $\mathcal{R}$, then the homology $H_q(X;M) = H_q(X;\mathcal{R})$.

Theorem 4.5.3 (Universal coefficient theorem). *Let C_* be a chain complex of free $\mathbb{Z}$-module and M a $\mathbb{Z}$-module. Then*

$$H_q(C_* \otimes_{\mathbb{Z}} M) \cong H_q(C_*) \otimes_{\mathbb{Z}} M \bigoplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M). \tag{4.5.1}$$

Example 4.5.4. 1. Recall that in Example 4.1.5 we computed $H_q(\mathbb{RP}^2;\mathbb{F}_2)$ without referring to $H_i(\mathbb{RP}^2;\mathbb{Z})$. Let's excercise Theorem 4.5.3 to compute $H_2(\mathbb{RP}^2;\mathbb{F}_2)$ using $H_i(\mathbb{RP}^2;\mathbb{Z})$. In fact,

$$\begin{aligned} H_2(\mathbb{RP}^2;\mathbb{F}_2) &\cong H_2(\mathbb{RP}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \bigoplus \mathrm{Tor}_1(H_1(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2) \\ &\cong 0 \otimes_{\mathbb{Z}} \mathbb{F}_2 \bigoplus \mathrm{Tor}_1(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong 0 \oplus \mathbb{F}_2 \end{aligned}$$

where we used (4.4.3) in the last line.

2. Similarly,

$$\begin{aligned} H_2(\mathbb{S}^2;\mathbb{F}_2) &\cong H_2(\mathbb{S}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \bigoplus \mathrm{Tor}_1(H_1(\mathbb{S}^2;\mathbb{Z}), \mathbb{F}_2) \\ &\cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_2 \bigoplus \mathrm{Tor}_1(0, \mathbb{F}_2) \\ &\cong \mathbb{F}_2 \oplus 0. \end{aligned}$$

Remark 4.5.5. The isomorphism in (4.5.1) is not natural. For instance, consider the quotient map $\mathbb{S}^2 \to \mathbb{RP}^2$. Although $H_2(\mathbb{S}^2;\mathbb{F}_2) \cong H_2(\mathbb{RP}^2, \mathbb{F}_2)$ is an isomorphism of $\mathbb{F}_2$-modules, the isomorphism does not send $\mathrm{Tor}_1(H_1(\mathbb{S}^2;\mathbb{Z}), \mathbb{F}_2)$ to $\mathrm{Tor}_1(H_1(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2)$ and $H_2(\mathbb{S}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$ to $H_2(\mathbb{RP}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$.

Proof of Theorem 4.5.3. Suppose C_* is a chain complex of $\mathbb{Z}$-modules and M is *a* $\mathbb{Z}$-module. For simplicity, we will omit the subscript of $\otimes_{\mathbb{Z}}$. If M is free then $H_q(C_* \otimes M) \cong H_q(C_*) \otimes M$. On the other hand, $\mathrm{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M)$ vanishes because M is free. Therefore, (4.5.1) holds. In general, we can find a free resolution F_* of M (cf. Remark 4.3.6 1) as follows:

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

In fact, we can take F_* such that $0 \to F_1 \to F_0 \to M \to 0$ is exact. Since C_* is free we get an exact sequence of chain complexes $0 \to C_* \otimes F_1 \to C_* \otimes F_0 \to C_* \otimes M \to 0$ (cf. Remark 4.2.17). Then the Snake Lemma 2.3.15 implies that there is a long exact sequence

$$\begin{tikzcd}
{} \arrow[r, dotted] & H_q(C_* \otimes F_1) \arrow[r] & H_q(C_* \otimes F_0) \arrow[r] & H_q(C_* \otimes M) \arrow[dll, rounded corners, to path={[pos=0.5] -- ([xshift=2ex]\tikztostart.east) |- node[above]{∂} ([yshift=1ex]\tikztotarget.north) -- (\tikztotarget)}] \\
& H_{q-1}(C_* \otimes F_1) \arrow[r] & H_{q-1}(C_* \otimes F_0) \arrow[r] & H_{q-1}(C_* \otimes M) \arrow[r, dotted] & {} .
\end{tikzcd}$$

Unpacking this long exact sequence, we get a family of short exact sequences

$$0 \to H_q(C_* \otimes F_0)/H_q(C_* \otimes F_1) \to H_q(C_* \otimes M) \to \ker(H_{q-1}(C_* \otimes F_1) \to H_{q-1}(C_* \otimes F_0)) \to 0.$$

Since C_* is free we can write $H_q(C_* \otimes F_0)/H_q(C_* \otimes F_1) \cong H_q(C_*) \otimes M$. Further,

$$\begin{aligned}
\ker(H_{q-1}(C_* \otimes F_1) \to H_{q-1}(C_* \otimes F_0)) &= \ker(H_{q-1}(C_*) \otimes F_1 \to H_{q-1}(C_*) \otimes F_2) \\
&\cong H_1(H_{q-1}(C_*) \otimes^{\mathbb{L}} M) \\
&\cong \mathrm{Tor}_1(H_{q-1}(C_*), M).
\end{aligned}$$

Therefore, we get a natural exact sequence

$$0 \to H_q(C_*) \otimes_{\mathbb{Z}} M \to H_q(C_* \otimes_{\mathbb{Z}} M) \to \mathrm{Tor}_1(H_{q-1}(C_*), M) \to 0. \tag{4.5.2}$$

We leave it for the reader to prove (4.5.1). □

Example 4.5.6. 1. Using (4.5.2), we know that the quotient map $\mathbb{S}^2 \to \mathbb{RP}^2$ induces the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2(\mathbb{S}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \arrow[r] \arrow[d] & H_2(\mathbb{S}^2;\mathbb{F}_2) \arrow[r] \arrow[d] & \mathrm{Tor}_1(H_1(\mathbb{S}^2;\mathbb{Z}),\mathbb{F}_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(\mathbb{RP}^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \arrow[r] & H_2(\mathbb{RP}^2;\mathbb{F}_2) \arrow[r] & \mathrm{Tor}_1(H_1(\mathbb{RP}^2;\mathbb{Z}),\mathbb{F}_2) \arrow[r] & 0
\end{tikzcd}$$

that is the same as

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_2 \arrow[r, "\cong"] \arrow[d] & \mathbb{F}_2 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{F}_2 \arrow[r, "\cong"] & \mathbb{F}_2 \arrow[r] & 0.
\end{tikzcd}$$

2. Similarly, the pinch map $\mathbb{RP}^2 \to \mathbb{S}^2$ that identifies $Sk_1\mathbb{RP}^2$ to a point induces the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{F}_2 \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathbb{F}_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{F}_2 \arrow[r, "\cong"] & \mathbb{F}_2 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

In general, we have the following version of Theorem 4.5.3 that gives a complete answer to the Question 4.0.1.

Theorem 4.5.7 (Universal coefficient theorem 2.0/Kunneth Theorem). *Let $\mathcal{R}$ be a principal ideal domain (PID). Suppose C_* is a chain complex of free $\mathcal{R}$-modules and D_* is a chain complex of $\mathcal{R}$-modules. Then there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes_{\mathcal{R}} H_q(D_*) \to H_n(C_* \otimes_{\mathcal{R}} D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathcal{R}}(H_p(C_*), H_q(D_*)) \to 0. \tag{4.5.3}$$

Further, the sequence splits, i.e., the middle term is a direct sum of other terms, but not naturally.

Sketch of the proof. **Idea 1:** Replace D_* with an isomorphic object in $\mathcal{D}(\mathcal{R})$. In fact, set $D'_n \coloneqq H_q(D_*)$ and consider the chain complex D'_* where all the boundary maps are 0. Note that D'_* is quasi-isomorphic to D_*. Further, we can view D'_* as a direct sum of chain complex concentrated only at one degree. Finally, use Theorem 4.5.3 and with a little bit of work we will get (4.5.3).

Idea 2: This proof does not require $\mathcal{D}(\mathcal{R})$. Consider the chain complex $Z(D_*)$ such that $Z(D_*)_n \coloneqq \ker(D_n \to D_{n-1})$ and all boundary maps are 0. Since C_* is free there is an exact sequence

$$0 \to C_* \otimes_{\mathcal{R}} Z(D_*) \to C_* \otimes_{\mathcal{R}} D_* \to C_* \otimes_{\mathcal{R}} D/Z(D_*) \to 0.$$

Then (4.5.3) follows with a little more work. □

Remark 4.5.8. The Tor term in Theorem 4.5.7 vanishes if $\mathcal{R}$ is a field in which case we get an isomorphism

$$\bigoplus_{p+q=n} H_p(C_*) \otimes_{\mathcal{R}} H_q(D_*) \cong H_n(C_* \otimes_{\mathcal{R}} D_*). \tag{4.5.4}$$

4.6 Eilenberg–Zilber

The universal coefficient theorem relates the homology of tensor products of two chain complexes of free $\mathcal{R}$-modules with that of the individual chain complexes. In this section, we will prove a topological analog of the universal coefficient theorem that allows us to understand the homology of products of spaces in terms of the homology of individual spaces. In particular, we will prove the following theorem.

Theorem 4.6.1 (Eilenberg–Zilber). *Suppose $X, Y \in$ **Top** and $\mathcal{R}$ is a commutative ring. Then $S_*(X \times Y; \mathcal{R})$ is quasi-isomorphic to $S_*(X;\mathcal{R}) \otimes_{\mathcal{R}} S_*(Y;\mathcal{R})$, i.e., they have the same homology groups.*

In practice, we compute $S_*(X;\mathcal{R}) \otimes_{\mathcal{R}} S_*(Y;\mathcal{R})$ using Theorem 4.5.7. Let's see a concrete consequence of Theorem 4.6.1 before proving it.

Example 4.6.2. Suppose $X = Y = \mathbb{RP}^2$ and $\mathcal{R} = \mathbb{F}_2$. Since $\mathbb{F}_2$ is a field we get the following isomorphism

$$H_n(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \bigoplus_{p+q=n} H_p(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_q(\mathbb{RP}^2; \mathbb{F}_2) \tag{4.6.1}$$

which together with (4.1.1) implies that

$$H_q(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } q = 0, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & \text{if } q = 1, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & \text{if } q = 2, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & \text{if } q = 3, \\ \mathbb{F}_2 & \text{if } q = 4, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6.2}$$

Remark 4.6.3. Note that the homological data with $\mathbb{F}_2$ coefficients of both $\mathbb{RP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$ vanish after q that encodes their dimension, see (4.1.1) and (4.6.2). Further, $H_q(\bullet, \mathbb{F}_2)$ is symmetric on q. We will explain these seemingly coincidental properties in Chapter 6.

To prove Theorem 4.6.1, we will explicitly construct a quasi-isomorphism between $S_*(X \times Y; \mathcal{R})$ and $S_*(X;\mathcal{R}) \otimes_{\mathcal{R}} S_*(Y;\mathcal{R})$. For the forward direction, we define $A_n : S_n(X \times Y; \mathcal{R}) \to \bigoplus_{p+q=n} S_p(X;\mathcal{R}) \otimes_{\mathcal{R}} S_q(Y;\mathcal{R})$ by setting $A_n \coloneqq \sum_{p+q=n} A_{(p,q)}$ where

$$A_{(p,q)} : S_{p+q}(X \times Y; \mathcal{R}) \to S_p(X;\mathcal{R}) \otimes_{\mathcal{R}} S_q(Y;\mathcal{R})$$

are maps we need to define. Recall that $S_{p+q}(X \times Y; \mathcal{R})$ is a free $\mathcal{R}$-module generated by the maps $\sigma : \Delta^{p+q} \to X \times Y$ that can be written as (σ_1, σ_2) where $\sigma_1 : \Delta^{p+q} \to X$ and $\sigma_2 : \Delta^{p+q} \to Y$. A first guess is to send σ to tensor products of restrictions of σ_1 and σ_2 on Δ^p and Δ^q respectively. Precisely, we have the following map.

Definition 4.6.4. The *Alexander–Whitney map* is a chain map $A_* : S_*(X \times Y; \mathcal{R}) \to S_*(X;\mathcal{R}) \otimes_{\mathcal{R}} S_*(Y;\mathcal{R})$ such that for each n, all $p + q = n$ and $\sigma = (\sigma_1, \sigma_2) \in S_{p+q}(X \times Y; \mathcal{R})$ we have

$$A_{(p,q)}(\sigma) \coloneqq (\sigma_1|_{\Delta^p}) \otimes (\sigma_2|_{\Delta^q}).$$

Here, $|_{\Delta^p}$ is the restriction to Δ^p given by $(e_0, \dots e_p) \mapsto (e_0, \dots, e_p, 0 \dots 0)$ and $|_{\Delta^q}$ is the restriction to Δ^q given by $(e_0, \dots e_q) \mapsto (0 \dots 0, e_0, \dots, e_q)$.

Remark 4.6.5. The Alexander–Whitney map induces the "obvious" isomorphism $H_0(X \times Y; \mathcal{R}) \to H_0(X;\mathcal{R}) \otimes_{\mathcal{R}} H_0(Y;\mathcal{R})$.

We will prove that A_* is a chain map using a generalized version of the Fundamental Theorem of Homological Algebra which we will prove below.

Denote $\mathcal{C} \coloneqq \mathbf{Fun}(\mathbf{Top} \times \mathbf{Top}; \mathcal{R}\text{-}\mathbf{mod})$ to mean the category of functors from $\mathbf{Top} \times \mathbf{Top}$ to $\mathcal{R}$-**mod** and $\mathbf{ch}(\mathcal{C}) \coloneqq \mathbf{Fun}(\mathbf{Top} \times \mathbf{Top}, \mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod}))$ to mean the category of functors from $\mathbf{Top} \times \mathbf{Top}$ to $\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})$. Further, define $\mathbb{M} \coloneqq \{(\Delta^p, \Delta^q) \mid p, q \geq 0\} \subset \mathrm{ob}(\mathbf{Top} \times \mathbf{Top})$.

Definition 4.6.6. A functor $F : \mathbf{Top} \times \mathbf{Top} \to \mathcal{R}\text{-}\mathbf{mod}$ is $\mathbb{M}$-*free* if there exists a collection $\{m_1, \dots m_l\} \subset \mathbb{M}$ such that

$$F((X,Y)) \cong \mathcal{R}\{\sqcup_i \mathbf{Hom}_{\mathbf{Top}\times\mathbf{Top}}(m_i, (X,Y))\}.$$

Example 4.6.7.
1. $S_n(X \times Y; \mathcal{R})$ is an $\mathbb{M}$-free functor since for $\{m_1, \dots m_l\} = \{(\Delta^n, \Delta^n)\}$ we have

$$\begin{aligned} S_n(X \times Y; \mathcal{R}) &= \mathcal{R}\{\mathbf{Hom}_{\mathbf{Top}}(\Delta^n, X \times Y)\} \\ &= \mathcal{R}\{\mathbf{Hom}_{\mathbf{Top}}(\Delta^n, X) \times \mathbf{Hom}_{\mathbf{Top}}(\Delta^n, Y)\} \\ &= \mathcal{R}\{\mathbf{Hom}_{\mathbf{Top}\times\mathbf{Top}}((\Delta^n, \Delta^n), (X,Y))\}. \end{aligned}$$

2. $S_n(X, \mathcal{R}) \otimes_{\mathcal{R}} S_n(Y; \mathcal{R})$ is also an $\mathbb{M}$-free functor where we take $\{m_i\}_i = \{(\Delta^{n-i}, \Delta^i)\}_{0 \leq i \leq n}$. In fact, since $S_p(X;\mathcal{R})$ and $S_q(Y;\mathcal{R})$ are free we get

$$\begin{aligned} S_n(X, \mathcal{R}) \otimes_{\mathcal{R}} S_n(Y; \mathcal{R}) &= \bigoplus_{p+q=n} S_p(X;\mathcal{R}) \otimes_{\mathcal{R}} S_q(Y;\mathcal{R}) \\ &= \bigoplus_{p+q=n} \mathcal{R}\{\mathbf{Sing}_p(X) \times \mathbf{Sing}_q(Y)\} \\ &= \mathcal{R}\{\sqcup_{p+q=n} \mathbf{Sing}_p(X) \times \mathbf{Sing}_q(Y)\} \\ &= \mathcal{R}\{\sqcup_{p+q=n} \mathbf{Hom}((\Delta^p, \Delta^q), (X,Y))\}. \end{aligned}$$

Remark 4.6.8. The $\mathbb{M}$-free functors depend only on simplicies.

Definition 4.6.9. An $\mathbb{M}$-*free resolution* of a functor $F : \mathbf{Top} \times \mathbf{Top} \to \mathcal{R}\text{-}\mathbf{mod}$ is a functor $F_* : \mathbf{Top} \times \mathbf{Top} \to \mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})$ with a natural transformation $\epsilon_F : H_0(F_*) \to F$ such that

- each F_n is $\mathbb{M}$-free, and
- for all $(\Delta^p, \Delta^q) \in \mathbb{M}$, $F_*((\Delta^p, \Delta^q))$ is a free resolution of $F(\Delta^p, \Delta^q)$.

In short, a free resolution F_* of F satisfies

$$H_i(F_*((\Delta^p, \Delta^q))) = \begin{cases} 0 & \text{if } i \neq 0, \\ F((\Delta^p, \Delta^q)) & \text{if } i = 0 \end{cases}$$

and $\epsilon_F : H_0(F_*((\Delta^p, \Delta^q))) \to F(\Delta^p, \Delta^q)$ is an isomorphism in $\mathcal{R}$-**mod**.

Example 4.6.10. 1. $S_*(X \times Y; \mathcal{R})$ is an $\mathbb{M}$-free resolution of $H_0(X \times Y; \mathcal{R})$.

2. $S_*(X; \mathcal{R}) \otimes_{\mathcal{R}} S_*(Y; \mathcal{R})$ is an $\mathbb{M}$-free resolution of $H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R})$.

Theorem 4.6.11 (Fundamental theorem of homological algebra 2.0). *Let $\Theta : F \to G$ be a natural transformation between functors F, G : $\mathbf{Top} \times \mathbf{Top} \to \mathcal{R}$-$\mathbf{mod}$. If F_*, G_* : $\mathbf{Top} \times \mathbf{Top} \to \mathbf{ch}(\mathcal{R}$-$\mathbf{mod})$ are $\mathbb{M}$-free resolutions of F and G respectively, then there exists a natural transformation $\Theta_* : F_* \to G_*$ such that $H_0(\Theta_*) = \Theta$. Further, Θ_* is unique up to natural chain homotopy.*

Proof. Same proof as that of Theorem 4.3.8. □

Proof of Theorem 4.6.1. Consider $S_*(X \times Y; \mathcal{R})$ as an $\mathbb{M}$-free resolution of $H_0(X \times Y; \mathcal{R})$ and $S_*(X; \mathcal{R}) \otimes_{\mathcal{R}} S_*(Y; \mathcal{R})$ as that of $H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R})$. Recall that we have an isomorphism $H_0(X \times Y; \mathcal{R}) \to H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R})$. In addition, the Alexander–Whitney map A_* : $S_*(X \times Y; \mathcal{R}) \to S_*(X; \mathcal{R}) \otimes_{\mathcal{R}} S_*(Y; \mathcal{R})$ lifts the natural map $H_0(X \times Y; \mathcal{R}) \to H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R})$. Choose a chain map $\Theta_* : S_*(X; \mathcal{R}) \otimes_{\mathcal{R}} S_*(Y; \mathcal{R}) \to S_*(X \times Y; \mathcal{R})$ lifting the inverse map $H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R}) \to H_0(X \times Y; \mathcal{R})$. Then $A_* \circ \Theta_*$ and $\Theta_* \circ A_*$ lift $\mathbb{1}_{H_0(X \times Y; \mathcal{R})}$ and $\mathbb{1}_{H_0(X; \mathcal{R}) \otimes_{\mathcal{R}} H_0(Y; \mathcal{R})}$ respectively. Therefore, Theorem 4.6.11 implies that $A_* \circ \Theta_*$ and $\Theta_* \circ A_*$ are chain-homotopic to the identity of respective spaces whence $S_*(X \times Y; \mathcal{R})$ is quasi-isomorphic to $S_*(X; \mathcal{R}) \otimes_{\mathcal{R}} S_*(Y; \mathcal{R})$. □

Remark 4.6.12. Define $H_*(X; \mathcal{R}) \coloneqq \bigoplus_q H_q(X; \mathcal{R})$. Instead of dealing with individual groups H_q, we can study $H_*(X)$ as an independent object. For instance, the Euler characteristic is an invariant associated with $H_*(X; \mathcal{R})$.

If the homology groups H_q are free then $H_*(X; \mathcal{R})$ has a co-multiplicative structure induced by then diagonal map $\Delta : X \to X \times X$ that sends $x \mapsto (x, x)$. In fact, Theorem 4.6.1 and Theorem 4.5.7 and the fact that Tor vanish since H_q's are free imply that

$$H_*(X; \mathcal{R}) \otimes_{\mathcal{R}} H_*(X; \mathcal{R}) \cong H_*(X \times X; \mathcal{R}).$$

In particular, Δ induces a *co-multiplication map* $H_*(X; \mathcal{R}) \to H_*(X; \mathcal{R}) \otimes_{\mathcal{R}} H_*(X; \mathcal{R})$. Here, co-multiplication can be thought of as a dual to the multiplication map defined below.

Definition 4.6.13. If A is an $\mathcal{R}$-module, a *multiplication* on A is an $\mathcal{R}$-module map m : $A \otimes_{\mathcal{R}} A \to A$ that satisfies

- $m(a_1 \otimes (a_2 + a_3)) = m(a_1 \otimes a_2) + m(a_1 \otimes a_3)$
- $m((a_1 + a_2) \otimes a_3) = m(a_1 \otimes a_3) + m(a_2 \otimes a_3)$.

A *comultiplication* is the dual map $A \to A \otimes_{\mathcal{R}} A$.

We will come back to multiplication in the next chapter.

Chapter 5

Cohomology

In this chapter, we will study *cohomology groups* H^*, the dual of homology groups. Cohomology groups retain the information of homology groups unless there are infinitely many cells in which case we lose the information by taking dual.[1] In fact, we have analogs (cf. Theorem 5.1.5) of Theorem 4.5.7. However, there is a multiplicative structure on H^* making it an associated, unital, graded commutative ring. The extra graded ring structure allows us to distinguish spaces that individual (co)homology groups can't (cf. Remark 5.3.10).

5.1 Homology dualized

Fix $X \in \mathbf{Top}$ and $M \in \mathbf{Ab}$.

Definition 5.1.1. 1. The *cochain complex, $S^*(X;M)$,* of X with coefficients in M is

$$S^n(X;M) \coloneqq \underline{\mathbf{Hom}}_{\mathbf{Ab}}(S_n(X;M), M)$$

with the boundary maps $\partial : S^n(X;M) \to S^{n+1}(X;M)$ defined by

$$(\partial f)\sigma \coloneqq (-1)^{n+1} f(\partial\sigma) \tag{5.1.1}$$

for any $f \in \underline{\mathbf{Hom}}_{\mathbf{Ab}}(S_n(X;M), M)$ and $\sigma \in S_{n+1}(X)$.

2. The *cohomology group* of $X \in \mathbf{Top}$ with coefficients in $M \in \mathbf{Ab}$ is $H^q(X;M) \coloneqq H_{-q}(S^*(X;M))$ where we view $S^*(X;M)$ as a chain complex where $S^n(X;M)$ is concentrated at degree $-n$.

Remark 5.1.2. 1. Note that $S^*(X;M) \cong \underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X;M), M)$ where M is viewed as a deg 0 chain complex. Here,

$$\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X;M), M)_n = \underline{\mathbf{Hom}}_{\mathbf{Ab}}(S_{-n}(X;M), M)$$

and differentials are linearly dual to differentials in $S_*(X;M)$.

[1] Taking dual is not a reflexive operation in infinite dimension.

2. [Hat05] defines the cochain complex $S^*(X;M)$ without the sign $(-1)^{n+1}$ in (5.1.1). However, both definitions are isomorphic in $\mathcal{D}(\mathbb{Z})$ because the kernel and image of a map are the same as that of the negatives of the maps. In particular, both definitions produce the same cohomology groups. However, our definition allows us to think $\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X), M)$ in terms of tensor products using the currying isomorphism.

3. For the purpose of computation of cohomology groups, we can and will use $C_*^{cell}(X;M)$ instead of $S_n(X;M)$ in the definition of $S^*(X;M)$.

4. Recall that for any $X \in \mathbf{Top}$, $H_0(X,M)$ measures the number of path components of X. Similarly, $H^0(X;M) = \{\text{functions from path components to } M\}$.

5. A map $f : X \to Y \in \mathbf{Top}$ induces a map of cohomology groups $H^*(Y;M) \to H^*(X;M)$. In particular, cohomology is a contravariant functor.

6. Cohomology functors satisfy the Eilenberg–Steenrod axioms. We also have Mayer–Vietoris theorem for cohomology groups.

Example 5.1.3. 1. Suppose $X = \mathbb{RP}^2$ and $M = \mathbb{Z}$. Then $C_*^{cell}(\mathbb{RP}^2;\mathbb{Z})$ is $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. Check that $S^*(\mathbb{RP}^2;\mathbb{Z})$ is $0 \to \mathbb{Z} \xrightarrow{-0} \mathbb{Z} \xrightarrow{+2} \mathbb{Z}$. Then

$$H^q(\mathbb{RP}^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

2. Similarly, $S^*(\mathbb{RP}^2;\mathbb{F}_2)$ is $\mathbb{F}_2 \xrightarrow{-0} \mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2$. Therefore,

$$H^q(\mathbb{RP}^2;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } q = 0, \\ \mathbb{F}_2 & \text{if } q = 1, \\ \mathbb{F}_2 & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We have an analog of the Universal Coefficient Theorem 4.5.3. The difference is we use Ext functor instead of Tor . Here, we define

$$\operatorname{Ext}^i_{\mathbb{Z}}(H_{q-1}(X), M) := H_{-i}(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(H_{q-1}(X), M)) \tag{5.1.2}$$

which is computed by replacing $H_{q-1}(X)$ with its free resolution.

Remark 5.1.4. In Remark 4.4.7, we alluded to an equivalent definition of Tor_i using the internal **Hom**. In fact, for any $M, N \in \mathcal{R}$-**mod** we can define

$$\operatorname{Tor}_i(M, N) := H^{-i}(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathcal{R})}(F_*, N)) \tag{5.1.3}$$

where F_* is a free resolution of M.

Theorem 5.1.5 (Universal coefficient theorem 3.0). *Suppose* $X \in \mathbf{Top}$ *and* $M \in \mathbf{Ab}$. *Then*

$$H^q(X;M) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(X;\mathbb{Z}), M) \oplus \underline{\mathbf{Hom}}_{\mathbf{Ab}}(H_q(X;\mathbb{Z}), M).$$

Example 5.1.6. We will recompute $H^q(\mathbb{RP}^2;\mathbb{F}_2)$ using Theorem 5.1.5 In fact,

$$\begin{aligned} H^1(\mathbb{RP}^2;\mathbb{F}_2) &= \mathrm{Ext}^1_{\mathbb{Z}}(H_0(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2) \oplus \underline{\mathbf{Hom}}_{\mathbf{Ab}}(H_1(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2) \\ &\cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_2) \oplus \underline{\mathbf{Hom}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2) \\ &\cong 0 \oplus \mathbb{F}_2. \end{aligned}$$

Note that $\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)$ is $\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}, \mathbb{F}_2)$ which is $0 \to \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2$. Therefore,

$$\begin{aligned} H^2(\mathbb{RP}^2;\mathbb{F}_2) &= \mathrm{Ext}^1_{\mathbb{Z}}(H_1(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2) \oplus \underline{\mathbf{Hom}}_{\mathbf{Ab}}(H_2(\mathbb{RP}^2;\mathbb{Z}), \mathbb{F}_2) \\ &\cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2) \oplus \underline{\mathbf{Hom}}(0, \mathbb{F}_2) \\ &\cong H_{-1}(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)) \oplus 0 \\ &\cong \mathbb{F}_2. \end{aligned}$$

Remark 5.1.7. We also have an analog of Theorem 4.6.1 that allows us to compute $H^q(X \times Y; M)$ in terms of $H_*(X;\mathbb{Z})$ and $H_*(Y;\mathbb{Z})$. In fact, when $H_*(X;\mathcal{R})$ and $H_*(Y;\mathcal{R})$ are free $\mathcal{R}$-modules then

$$H^*(X;\mathcal{R}) \otimes_{\mathcal{R}} H^*(Y \times \mathcal{R}) \cong H^*(X \times Y;\mathcal{R}). \tag{5.1.4}$$

5.2 Cup product

By abuse of notation, we define the graded abelian group $H^*(X;\mathcal{R})$ as

$$H^*(X;\mathcal{R}) := \bigoplus_{i \geq 0} H^i(X;\mathcal{R}).$$

We call an element $x \in H^q(X;\mathcal{R}) \subset H^*(X;\mathcal{R})$ *homogeneous of* $\deg q$. Further, we will write $|x| = q$. Here, every element in $H^*(X;\mathcal{R})$ is a sum of finitely many homogeneous elements. In this section, we will set up a mechanism to answer the following question:

Question 5.2.1. Can $H^*(X_i;\mathcal{R})$ be "different" even if $H^q(X_1;\mathcal{R}) = H^q(X_2;\mathcal{R})$ for all q?

To make sense of the question, we have to find an extra structure on $H^*(X;\mathcal{R})$. In fact, we will prove that it has a ring structure induced by a multiplication map (cf. Theorem 5.2.6). In any case, the main goal is to use $H^*(X;\mathcal{R})$ to distinguish spaces that individual cohomology groups can't.

Definition 5.2.2. Suppose $X, Y \in$ **Top** and $\mathcal{R}$ is a ring. The *cohomology cross product* $\times : H^*(X;\mathcal{R}) \otimes_{\mathcal{R}} H^*(Y;\mathcal{R}) \to H^*(X \times Y;\mathcal{R})$ is the composite of the following maps:

$$\begin{aligned} H^*(X;\mathcal{R}) \otimes_{\mathcal{R}} H^*(Y;\mathcal{R}) &\xrightarrow{f_1} H^*(S^*(X;\mathcal{R}) \otimes_{\mathcal{R}} S^*(Y;\mathcal{R})) \\ &\xrightarrow{f_2} H^*(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), \mathcal{R})) \\ &\xrightarrow{f_3} H^*(X \times Y;\mathcal{R}). \end{aligned}$$

Here, f_1 is the map as in the Kunneth theorem 4.5.7. The map exists even if $\mathcal{R}$ is not PID but we don't get a short exact sequence.

Further, f_2 is induced from the chain map

$$\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X), \mathcal{R}) \otimes_{\mathbb{Z}} \underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(Y), \mathcal{R}) \to \underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), \mathcal{R})$$

given explicitly by

$$f \otimes g \mapsto \left\{ x \otimes y \mapsto \begin{cases} (-1)^{pq} f(x) g(y) & \text{if } |x| = |f| = p \ \&\ |y| = |g| = q, \\ 0 & \text{otherwise.} \end{cases} \right.$$

Finally, f_3 is an isomorphism induced by Theorem 4.6.1, i.e.,

$$\begin{aligned} H^*(X \times Y;\mathcal{R}) &\cong H_{-*}(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X \times Y), \mathcal{R})) \\ &\cong H_{-*}(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), \mathcal{R})) \\ &= H^*(\underline{\mathbf{Hom}}_{\mathcal{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), \mathcal{R})). \end{aligned}$$

Remark 5.2.3.

1. The cross product $\times$ is an isomorphism of graded abelian groups if either $H_q(X;\mathcal{R})$ is free and finitely generated $\mathcal{R}$-module for all q or $H_q(Y;\mathcal{R})$ is free and finitely generated for all q.

Definition 5.2.4. Suppose $X \in$ **Top** and $\mathcal{R}$ is ring. The *cup product multiplication* $\cup$: $H^*(X;\mathcal{R}) \otimes_{\mathcal{R}} H^*(X;\mathcal{R}) \to H^*(X;\mathcal{R})$ is the composite of the maps:

$$H^*(X;\mathcal{R}) \otimes_{\mathcal{R}} H^*(X;\mathcal{R}) \xrightarrow{\times} H^*(X \times X;\mathcal{R}) \xrightarrow{H^*(\Delta)} H^*(X;\mathcal{R})$$

where $H^*(\Delta)$ is the map induced by the diagonal map Δ.

Remark 5.2.5.

1. To explicitly understand $\cup$, first note that any element $\bar{f} \in H^p(X;\mathcal{R})$ is an equivalence class represented by a co-cycle $f \in S^p(X;\mathcal{R}) = \underline{\mathbf{Hom}}_{\mathbf{Ab}}(S_p(X), \mathcal{R})$, i.e., $0 = (\partial f)\sigma = (-1)^{p+1} f(\partial\sigma)$ for every $\sigma \in S_{p+1}(X)$. Suppose $f : \mathbf{Sing}_p(X) \to \mathcal{R}$ and $g : \mathbf{Sing}_q(X) \to \mathcal{R}$ are two cycles representing $\bar{f}$ and $\bar{g}$ respectively. Then the cup product $\bar{f}\bar{g}$ is represented by $fg : \mathbf{Sing}_{p+q}(X) \to \mathcal{R}$ given by the formula

$$(fg)(\sigma) = (-1)^{pq} f(\sigma|_{\Delta^p}) g(\sigma|_{\Delta^q}) \tag{5.2.1}$$

where $|_{\Delta^p}$ is restriction to the "front p-face" and the $|_{\Delta^q}$ is restriction to the "back q-face" of Δ^{p+q} similar to that in Alexander–Whitney map.

2. The cup product is the wedge product in the case of de Rham cohomology.

3. By abuse of notation, we will write $\cup(x \otimes y) = xy$.

Theorem 5.2.6. *For any $X \in$ **Top** and a commutative ring $\mathcal{R}$, $H^*(X;\mathcal{R})$ is a graded commutative ring, i.e.,*

- ***Closure:*** *If $x \in H^p(X;\mathcal{R})$ and $y \in H^p(Y;\mathcal{R})$, then $xy \in H^{p+q}(X;\mathcal{R})$.*
- ***Associativity:*** *For any $x, y, z \in H^*(X;\mathcal{R})$, $(xy)z = x(yz)$.*
- ***Identity:*** *There exists an element $\mathbb{1} \in H^0(X;\mathcal{R})$ such that $x\mathbb{1} = \mathbb{1}x = x$ for all $x \in H^*(X;\mathcal{R})$. Here $\mathbb{1} \in H^0(X,\mathcal{R})$ is the function that sends all path components to $\mathbb{1}_{\mathcal{R}} \in \mathcal{R}$.*
- ***Graded commutativity*** *If $x \in H^p(X;\mathcal{R})$ and $y \in H^q(X;\mathcal{R})$ then $xy = (-1)^{pq}yx$.*

Proof. We will prove associativity and leave the rest of the proof for the reader. Suppose $\bar{f} \in H^p(X;\mathcal{R})$, $\bar{g} \in H^q(X;\mathcal{R})$ and $\bar{h} \in H^r(X;\mathcal{R})$. Then for any $\sigma \in \mathbf{Sing}_{p+q+r}(X)$

$$\begin{aligned}((fg)h)\sigma &= (-1)^{(p+q)r}(fg)(\sigma|_{\Delta^{p+q}})h(\sigma|_{\Delta^r}) \\ &= (-1)^{pr+qr}(-1)^{pq}f(\sigma|_{\Delta^p})g(\sigma|_{\Delta^q})h(\sigma|_{\Delta^r}).\end{aligned}$$

Similarly,

$$\begin{aligned}(f(gh))\sigma &= (-1)^{p(q+r)}f(\sigma|_{\Delta^p})(gh)(\sigma|_{\Delta^{q+r}}) \\ &= (-1)^{pr+qr}(-1)^{pq}f(\sigma|_{\Delta^p})g(\sigma|_{\Delta^q})h(\sigma|_{\Delta^r}).\end{aligned}$$

□

Remark 5.2.7. 1. In the next chapter, we will give a geometric interpretation to the ring structure of $H^*(X;\mathcal{R})$ with $\cup$ as multiplication which will allow us to compute $\cup$ easily.

2. There is a cohomology ring functor H^*.

5.3 Cohomology ring

In this section, we will provide partial answers to the analog of Question 2.0.1 for the cohomology ring functor (cf. Theorems 5.3.4, 5.3.5 and 5.3.6). Further, we will see some examples of cohomology rings, mostly polynomial rings. Finally, we will see that cohomology rings are more powerful invariants than (co)homology groups, i.e., they distinguish spaces that (co)homology groups can't (cf. Remark 5.3.10).

Theorem 5.3.1. *The cohomology ring of $\mathbb{S}^n$, $n \geq 1$, is the polynomial ring $\mathbb{F}_2[w]/w^2$.*

Proof. Take the cellular structure on $\mathbb{S}^n$

$$\underbrace{\mathbb{F}_2}_{n} \to 0 \dashrightarrow 0 \to \mathbb{F}_2. \tag{5.3.1}$$

Then $H^0(\mathbb{S};\mathbb{F}_2) = \mathbb{F}_2\{1\}$, $H^n(\mathbb{S}^2,\mathbb{F}_2) = \mathbb{F}_2\{w\}$ and 0 otherwise. Therefore, $H^*(\mathbb{S}^2;\mathbb{F}_2) = \mathbb{F}_2[w]/w^2$. $\square$

Theorem 5.3.2. *Suppose K is the Klein bottle. Then*

$$H^*(K;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha,\beta,k]/(\alpha^2 = k, \beta^2, \alpha\beta = k, k\alpha, k\beta, k^2) \tag{5.3.2}$$

where $|\alpha| = |\beta| = 1$ and $|k| = 2$.

Proof. Consider a CW-structure on K as shown in Figure 5.1. Then the cellular chain

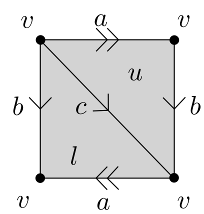


Figure 5.1: Klein bottle

complex is $C_*^{cell}(K;\mathbb{F}_2)$ is $0 \to \mathbb{F}_2\{u,l\} \to \mathbb{F}_2\{a,b,c\} \to \mathbb{F}_2\{v\} \to 0$. Therefore, $S^*(K;\mathbb{F}_2)$ is

$$0 \to \mathbb{F}_2\{\delta_v\} \to \mathbb{F}_2\{\delta_a,\delta_b,\delta_c\} \to \mathbb{F}_2\{\delta_u,\delta_l\} \to 0,$$

where $\delta_x(x) = 1$ and 0 otherwise. To compute the cohomology groups, we have to understand the boundary maps of $S^*(K;\mathbb{F}_2)$. First, note that $\partial\delta_v(a) = \partial\delta_v(b) = \partial\delta_v(c) = 0$. Second, $\partial(\delta_a)(u) = \delta_a(\partial u) = \delta_a(b - c + a) = 1$. Similarly, $\partial(\delta_a)(l) = 1$. Therefore, $\partial\delta_a = \delta_u + \delta_l$. In fact, $\partial\delta_b = \partial\delta_c$. Hence,

$$H^i(K;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2\{\delta_v\} & \text{if } i = 0, \\ \mathbb{F}_2\{\delta_a + \delta_b, \delta_b + \delta_c\} & \text{if } i = 1, \\ \mathbb{F}_2\{\delta_u\} & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.3}$$

It remains to understand the ring structure on $H^*(K;\mathbb{F}_2)$. For simplicity, we label $\alpha \coloneqq \delta_a + \delta_b$, $\beta \coloneqq \delta_b + \delta_c$ and $k = \delta_u$.

First, $\alpha^2 = k$. In fact, by ordering the vertices of u so that the edges point from lower order to a higher order, we get

$$((\delta_a + \delta_b)(\delta_a + \delta_b))(u) = (\delta_a + \delta_b)(a)(\delta_a + \delta_b)(b) = 1$$
$$((\delta_a + \delta_b)(\delta_a + \delta_b))(l) = (\delta_a + \delta_b)(c)(\delta_a + \delta_b)(a) = 0.$$

Similarly, $\beta^2 = 0$ and $\alpha\beta = k$. Further, $k^2 = 0$ since $H^4(K;\mathbb{F}_2) = 0$. Further, $\alpha k = \beta k = 0$ since $H^3(K;\mathbb{F}_2) = 0$. Check that δ_v is the identity element. Therefore, we see that $H^*(K;\mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[\alpha,\beta,k]/(\alpha^2 = k, \beta^2, \alpha\beta = k, k\alpha, k\beta, k^2)$. $\square$

As an application of the cohomological ring, let's understand the maps $f : \mathbb{S}^2 \to K$.

Proposition 5.3.3. *Any map $f : \mathbb{S}^2 \to K$ induces a trivial map of homology groups $H^2(K; \mathbb{F}_2) \to H^2(\mathbb{S}^2; \mathbb{F}_2)$.*

Proof. Note that f induces a map of cohomology rings

$$H^*(f) : \mathbb{F}_2[\alpha, \beta, k]/(\alpha^2 = k, \beta^2, \alpha\beta = k, k\alpha, k\beta, k^2) \to \mathbb{F}_2[\omega]/\omega^2 \tag{5.3.4}$$

Note that $|\alpha| = 1$, but there is no element of degree 1 in $H^*(\mathbb{S}^2; \mathbb{F}_2)$. Therefore, $H^*(f)(\alpha) = 0$. Since $k = \alpha^2$, $H^*(f)(k) = 0$, i.e., $H^2(f) \equiv 0$. □

Let's get back to answering the analog of Question 2.0.1.

Theorem 5.3.4. *Fix a commutative ring $\mathcal{R}$. For any $X, Y \in$ **Top** there is an isomorphism of cohomology rings*

$$H^*(X \sqcup Y, \mathcal{R}) \cong H^*(X, \mathcal{R}) \oplus H^*(Y, \mathcal{R})$$

where $(a, b) \cdot (a', b') = (aa', bb')$.

Proof. The inclusion maps $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ induce maps of graded cohomology rings given by $H^*(X \sqcup Y; \mathcal{R}) \xrightarrow{g_1} H^*(X)$ and $H^*(X \sqcup Y; \mathcal{R}) \xrightarrow{g_2} H^*(Y)$. Then g_i induce a map $H^*(X \sqcup Y; \mathcal{R}) \xrightarrow{f} H^*(X; \mathcal{R}) \oplus H^*(Y; \mathcal{R})$ where $f(u) = (g_1(u), g_2(u))$. Check that f is an isomorphism of graded groups which amounts to proving that the map is a bijection. □

Theorem 5.3.5. *The quotient map $X \sqcup Y \to X \bigvee Y$ induces a map of cohomology rings $H^*(X \bigvee Y) \to H^*(X \sqcup Y)$ that is an isomorphism in positive degrees and an injection in* deg 0.

Theorem 5.3.6. *The cohomology cross product*

$$\times : H^*(X; \mathcal{R}) \otimes_{\mathcal{R}} H^*(Y; \mathcal{R}) \to H^*(X \times Y; \mathcal{R})$$

is a map of graded rings where $H^*(X; \mathcal{R}) \otimes_{\mathcal{R}} H^*(Y; \mathcal{R})$ has a multiplicative structure such that for any $a_i \otimes b_i \in H^*(X; \mathcal{R}) \otimes_{\mathcal{R}} H^*(Y; \mathcal{R})$ we have

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{|b_1||a_2|}((a_1 a_2) \otimes (b_1 b_2)).$$

Further, $\times$ is an isomorphism if $H_*(X; \mathcal{R})$ and $H_*(Y; \mathcal{R})$ are finitely generated torsion free $\mathcal{R}$-modules.

Theorem 5.3.7. *We have*

$$\begin{aligned} H^*(\mathbb{S}^1 \sqcup \mathbb{S}^2) &\cong \mathbb{Z}[x]/x^2 \oplus \mathbb{Z}[y]/y^2, \qquad |x| = 2,\ |y| = 1 \\ &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{at } \deg 0, \\ \mathbb{Z}\{y\} & \text{at } \deg 1, \\ \mathbb{Z}\{x\} & \text{at } \deg 2, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

and

$$H^*(\mathbb{S}^1 \bigvee \mathbb{S}^2) \cong \begin{cases} \mathbb{Z}\{1\} & \textit{at } \deg 0, \\ \mathbb{Z}\{y\} & \textit{at } \deg 1, \\ \mathbb{Z}\{x\} & \textit{at } \deg 2. \end{cases}$$

Theorem 5.3.8. *The cohomology ring of $\mathbb{T}^2$ is*

$$H^*(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta,\gamma]/(\alpha^2, \alpha\beta = \gamma, \beta^2, \alpha\gamma, \gamma^2, \beta\gamma). \tag{5.3.5}$$

Proof. Recall that $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and $H^*(\mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}[y]/y^2$ with $|y| = 1$. Therefore, Theorem 5.3.6 implies that

$$H^1(\mathbb{T}^2;\mathbb{Z}) \cong H^*(\mathbb{S}^1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(\mathbb{S}^1;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\otimes 1\} & \text{at } \deg 0, \\ \mathbb{Z}\{y\otimes 1, 1\otimes y\} & \text{at } \deg 1, \\ \mathbb{Z}\{y \otimes y\} & \text{at } \deg 2, \\ 0 & \text{otherwise.} \end{cases}$$

Writing $\alpha = y \otimes 1$, $\beta = 1 \otimes y$ and $\gamma = y \otimes y$, check that

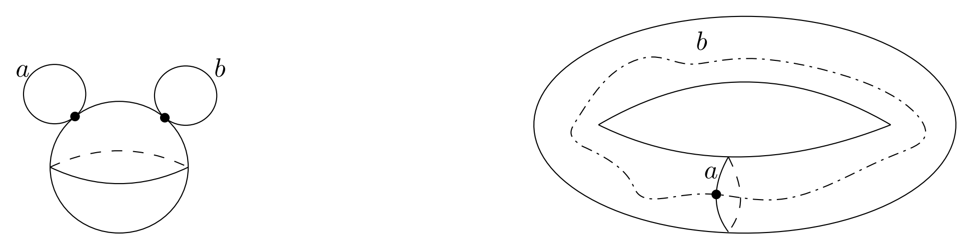
$$H^*(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta,\gamma]/(\alpha^2, \alpha\beta = \gamma, \beta^2, \alpha\gamma, \gamma^2, \beta\gamma). \tag{5.3.6}$$

□

Theorem 5.3.9. *Let $M \coloneqq (\mathbb{S}^2 \bigvee \mathbb{S}^1) \bigvee \mathbb{S}^1$, see Figure 5.2a. Then*

$$H^*(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{1\} & \textit{at } \deg 0, \\ \mathbb{Z}\{y_1, y_2\} & \textit{at } \deg 1, \\ \mathbb{Z}\{x\} & \textit{at } \deg 2, \\ 0 & \textit{otherwise,} \end{cases}$$

where $y_1^2 = y_2^2 = y_1 y_2 = 0$. In particular, $H^*(M;\mathbb{Z}) = \mathbb{Z}[x, y_1, y_2]/(\text{all products})$.



(a) Mickey mouse (b) $\mathbb{T}^2$

Figure 5.2: Non-homotopic spaces with same (co)homology groups

Remark 5.3.10. Observe that the Mickey mouse M and $\mathbb{T}^2$ are indistinguishable via cohomology and homology groups. However, the spaces are not homotopy equivalent since their cohomology rings are different. Namely, all products vanish in $H^*(M;\mathbb{Z})$, but $\alpha\beta$ does not vanish in $H^*(\mathbb{T};\mathbb{Z})$.

Intuitively, the spaces are non-homotopic since if we remove a point of M where a is glued to $\mathbb{S}^2$ then we get two connected components. However, if we remove any point in $\mathbb{T}^2$, we get just one connected component.

Algebraically, it means that although the pairs M, $M \times M$ and $\mathbb{T}^2$, $\mathbb{T}^2 \times \mathbb{T}^2$ have the same homology and cohomology groups, the diagonal maps $M \to M \times M$ and $\mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ do not induce the same map in homology.

A geometric reasoning is that, in M, the generators at degree 1, a, b, don't intersect, see Figure 5.2a. However, in $\mathbb{T}^2$, the generators at degree 1, a, b, intersect, see Figure 5.2b. In the next chapter, we will formalize how intersection plays a role in the geometric interpretation of cohomology rings.

Chapter 6

Poincaré duality

The main goal we set out at the beginning of these notes was to distinguish objects in **Top**. The first attempt we made was to probe $X \in \mathbf{Top}$ using functions from model spaces (simplices), therefore associating to X an object in $\mathbf{ch}(\mathcal{R}\text{-}\mathbf{mod})$. From the corresponding singular chain complex, we extracted algebraic invariant, (co)homology, which turns out to factor through $\mathbf{Ho}(\mathbf{Top})$ (and $\mathcal{D}(\mathcal{R})$). For the matter of computation, we got around by using cellular chain complexes which encode the identical information as the singular chain complex. Moreover, we also saw how the invariants behave under operations on **Top**. In any case, the algebraic objects that we worked with had topological analogs or interpretations. However, in the previous chapter, we saw that cohomology groups can be bagged together to form a ring, giving rise to a stronger invariant. In this chapter, we will answer the following questions:

Question 6.0.1. Is there a geometric (topological) interpretation of the ring structure on cohomology groups?

The natural models of topological spaces to seek the answer to Question 6.0.1 are manifolds (see Definition 6.1.1) which is what we will focus on for the rest of this chapter.

6.1 Perfect pairing

In this section, we will address the symmetry of (co)homological data we observed in Remark 4.6.3. The answer (cf. Theorem 6.1.4) will also give a geometric interpretation of the cohomological ring structure.

Definition 6.1.1. An *n-dimensional manifold M* is a Hausdorff topological space such that for all points $x \in M$ there exists an open neighborhood U of x homoemorphic to $\mathbb{R}^n$.

Example 6.1.2. 1. $\mathbb{R}^n, \mathbb{S}^n, \mathbb{RP}^n$ are n-dimensional manifolds.

2. $\mathbb{T}^n := \mathbb{S}^n \times \mathbb{S}^n$ is a $2n$-dimensional manifold.
3. The Klein bottle is a 2 dimensional manifold.
4. $\mathbb{S}^2 \bigvee \mathbb{S}^2$ is not a manifold.

Remark 6.1.3 (Geometric topology)**.** Every compact manifold is homotopy equivalent to a finite type CW complex.

Suppose X is a finite type CW complex. Then Theorem 5.1.5 gives a canonical isomorphism $H^p(X;\mathbb{F}_2) \cong \underline{\mathbf{Hom}}_{\mathbb{F}_2\text{-mod}}(H_p(X;\mathbb{F}_2),\mathbb{F}_2)$. By using a currying isomorphism, we view the duality of homology and cohomology via the map known as the *Kronecker pairing*:

$$\langle \bullet, \bullet \rangle : H_p(X;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^p(X;\mathbb{F}_2) \to \mathbb{F}_2. \tag{6.1.1}$$

In general, a *perfect pairing* between $\mathbb{F}_2$-modules V, W is a map $V \otimes_{\mathbb{F}_2} W \to \mathbb{F}_2$ such that the curried map $V \to \underline{\mathbf{Hom}}(W, \mathbb{F}_2)$ is an isomorphism.

When it comes to compact manifolds, we get duality via a perfect pairing within cohomology groups in addition to the duality of homology and cohomology. As we will see soon, the duality within cohomology groups explains the symmetry of cohomology groups.

Theorem 6.1.4 (Poincaré duality I). *Let M be a compact n-dimensional manifold. Then there is a fundamental class $[M] \in H_n(M;\mathbb{F}_2)$ such that for all p, q with $p + q = n$, the composite*

$$H^p(M;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^q(M;\mathbb{F}_2) \xrightarrow{\cup} H^n(M;\mathbb{F}_2) \xrightarrow{\langle \bullet, [M] \rangle} \mathbb{F}_2$$

is a perfect pairing. In particular, $H^p(M;\mathbb{F}_2) \cong H^q(M;\mathbb{F}_2)^\vee$, where $()^\vee$ is the dual functor.

We will postpone the proof of Theorem 6.1.4 until §6.6 where we will state a general version and sketch out the proof. Meanwhile, let's see an example of the perfect pairing.

Example 6.1.5. Recall that

$$H^*(\mathbb{T}^2;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha,\beta,\gamma]/(\alpha^2, \alpha\beta = \gamma, \beta^2, \alpha\gamma, \gamma^2, \beta\gamma),$$
$$H_*(\mathbb{T}^2;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 \\ \mathbb{F}_2\{\delta_\alpha, \delta_\beta\} \\ \mathbb{F}_2\{\delta_\gamma\}. \end{cases}$$

Here, the fundamental class $[\mathbb{T}^2]$ is $\delta_\gamma \in H_2(M;\mathbb{F}_2)$. Then the perfect pairing

$$H^1(M;\mathbb{F}_2) \otimes H^1(M;\mathbb{F}_2) \to \mathbb{F}_2$$

can be explicitly computed as:

$$\begin{aligned} \alpha \otimes \alpha &\mapsto \delta_\gamma(\alpha^2) = \delta_\gamma(0) = 0 \\ \alpha \otimes \beta &\mapsto \delta_\gamma(\alpha\beta) = \delta_\gamma(\gamma) = 1 \\ \beta \otimes \alpha &\mapsto \delta_\gamma(\beta\alpha) = 1 \\ \beta \otimes \beta &\mapsto \delta_\gamma(\beta^2) = 0. \end{aligned}$$

Before proving Theorem 6.1.4 let's see its applications.

Proposition 6.1.6. *Suppose M is a compact 3-dimensional manifold. Assume that $H^0(M;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ and $H^1(M;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$. Then*

$$H_q(M;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & \textit{if } q = 0, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & \textit{if } q = 1, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & \textit{if } q = 2, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & \textit{if } q = 3, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The Poincaré duality and Kronecker pairing imply that $H^1(M;\mathbb{F}_2) \cong H^2(M;\mathbb{F}_2)^\vee \cong H_2(M;\mathbb{F}_2)$. □

Theorem 6.1.7. *The cohomological ring of $\mathbb{RP}^2$ is $H^*(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2\{x,y\}/(x^2-y, xy, y^2)$.*

Proof. Recall that

$$H^q(\mathbb{RP}^2;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{if } q = 0, \\ \mathbb{F}_2\{x\} & \text{if } q = 1, \\ \mathbb{F}_2\{y\} & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then Poincaré duality implies that there is a perfect pairing

$$H^1(\mathbb{RP}^2;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(\mathbb{RP}^2;\mathbb{F}_2) \xrightarrow{\cup} H^2(\mathbb{RP}^2;\mathbb{F}_2) \to \mathbb{F}_2$$

which forces that $x^2 = y$. □

Remark 6.1.8. Note that $\mathbb{RP}^2$ and $\mathbb{S}^2 \vee \mathbb{S}^1$ have the same (co)homology groups with $\mathbb{F}_2$ coefficients. However, their ring structures differ.

Theorem 6.1.9. *The cohomological ring of $\mathbb{RP}^3$ is $H^*(\mathbb{RP}^3;\mathbb{F}_2) \cong \mathbb{F}_2\{x\}/x^4$.*

Proof. Recall that

$$H^*(\mathbb{RP}^3;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{at } \deg 0, \\ \mathbb{F}_2\{x\} & \text{at } \deg 1, \\ \mathbb{F}_2\{y\} & \text{at } \deg 2, \\ \mathbb{F}_2\{z\} & \text{at } \deg 3, \\ 0 & \text{otherwise.} \end{cases}$$

To compute the ring structure, note that the inclusion $\mathbb{S}^2 \to \mathbb{S}^3$ induces a map of quotient space $\mathbb{RP}^2 \to \mathbb{RP}^3$ that induces a map of cohomology rings $H^*(\mathbb{RP}^3;\mathbb{F}_2) \to H^*(\mathbb{RP}^2;\mathbb{F}_2)$. Here, $x \mapsto x$ and $y \mapsto y$ and $z \mapsto 0$. Since $x^2 = y$ in $H^*(\mathbb{RP}^2;\mathbb{F}_2)$, we also get $x^2 = y$ in $H^*(\mathbb{RP}^3;\mathbb{F}_2)$. Therefore, $x^3 = xy$. On the other hand, $H^1(\mathbb{RP}^3;\mathbb{F}_2) \otimes H^2(\mathbb{RP}^3;\mathbb{F}_2) \to H^3(\mathbb{RP}^3;\mathbb{F}_2) \to \mathbb{F}_2$ is a perfect pairing which forces $x^3 = z$. □

In general we have the following theorem:

Theorem 6.1.10. *The cohomology ring of the real projective space is* $H^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}\{x\}/x^{n+1}$.

Theorem 6.1.11 (Borsuk–Ulam). *For all functions,* $f : \mathbb{S}^n \to \mathbb{R}^n$ *there exists some* $x \in \mathbb{S}^n$ *such that* $f(x) = f(-x)$.[1]

Proof. For simplicity, let $n = 2$. Suppose the theorem is false. Consider $g : \mathbb{S}^2 \to \mathbb{S}^1$ that sends $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Note that g is odd, i.e., $-g(x) = g(-x)$. Therefore, it induces a map $\overline{g} : \mathbb{RP}^2 \to \mathbb{RP}^1$ and a commuting square:

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "g"] \arrow[d, "q"] & \mathbb{S}^1 \arrow[d, "q"] \\
\mathbb{RP}^2 \arrow[r, "\overline{g}"] & \mathbb{RP}^1
\end{tikzcd}$$

where q is the quotient map. Applying the functor $H^1(\bullet;\mathbb{F}_2)$, we get a commuting diagram

$$\begin{tikzcd}
\mathbb{F}_2 & \mathbb{F}_2 \arrow[l] \\
\mathbb{F}_2 \arrow[u] & \mathbb{F}_2. \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd}$$

Using the covering space theory,[2] we see that the bottom map is an isomorphism. On the other hand, we also have a map of cohomological rings $\mathbb{F}_2[x]/x^3 \leftarrow \mathbb{F}_2[x]/x^2$ that sends $0 \neq x^2 \leftarrow\!\shortmid x^2 = 0$, a contradiction. $\square$

Remark 6.1.12. In one dimensional case, Borsuk–Ulam theorem follows from the intermediate value theorem.

6.2 Geometric interpretation of the Poincaré duality

Let M be an n-dimensional manifold and $p + q = n$. Using the Poincaré duality (PD) and the Kronecker pairing (KP) we get a canonical isomorphism

$$\underline{\mathbf{Hom}}_{\mathbb{F}_2}(H_p(M;\mathbb{F}_2),\mathbb{F}_2) \cong H^p(M;\mathbb{F}_2) \xrightarrow[\cong]{PD} \underline{\mathbf{Hom}}_{\mathbb{F}_2}(H^q(M;\mathbb{F}_2),\mathbb{F}_2) \overset{KP}{\cong} H_q(M;\mathbb{F}_2). \tag{6.2.1}$$

Question 6.2.1. What is the geometric meaning of the isomorphism (6.2.1)?

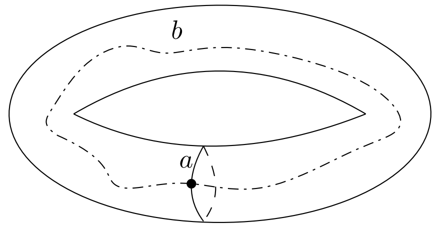


Figure 6.1: $\mathbb{T}^2$

In this section, we will answer the above question.

For simplicity, set $M = \mathbb{T}^2$. Then (6.2.1) implies that

$$H_1(\mathbb{T}^2; \mathbb{F}_2) \cong \underline{\mathbf{Hom}}_{\mathbb{F}_2}(H_1(\mathbb{T}^2; \mathbb{F}_2), \mathbb{F}_2). \tag{6.2.2}$$

Let $[a], [b]$ be the generators of $H_1(\mathbb{T}^2; \mathbb{F}_2)$, see Figure 6.1. Then the isomorphism (6.2.2) sends $[a] \mapsto f \in \underline{\mathbf{Hom}}_{\mathbb{F}_2}(H_1(\mathbb{T}^2; \mathbb{F}_2), \mathbb{F}_2)$ such that for any cycle c, $f(c)$ is the number of intersection of a with c. For instance, $f(b) = 1$ in Figure 6.1.

However, a cycle is defined up to the boundary, so we can perturb b to be curled near a, see 6.2. When the intersection looks like the one in the middle $f(b) = 3$. Since $3 = 1$ in $\mathbb{F}_2$, the map still makes sense. On the other hand, when b is tangent at one intersection, see Figure 6.2 (right) we have $f(b) = 0$. However, such b is unstable which means that if we perturb b, the number of intersections changes discontinuously. Therefore, $a \mapsto f$ such that f measures the number of intersection with a works for "generic" b.

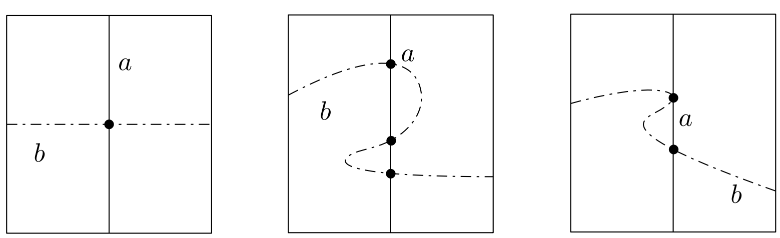


Figure 6.2: Intersection of a and b

In general, the isomorphism in (6.2.1) sends a p-dimensional cycle to a function that counts the intersection of p with generic q- dimensional cycles. Note that we view the intersection of cycles locally. Further, we have to consider smooth geometry so that we can formalize transverse intersections. However, since we only care about spaces up to homotopy type, it is be better if the intersection theory is defined for spaces other than smooth manifolds because transversality is hard to define in such spaces, for instance the Mickey mouse. In the following section, we will develop an intersection theory using homology but which still retains the geometric information of intersections.

[1]Physically, it means that there are two points, antipodal to one another on the Earth, with the same temperature and air pressure.

[2]The equator of $\mathbb{S}^2$ under g loops around $\mathbb{S}^1$.

6.3 Covering spaces

A natural question to ask after Question (6.2.1) is the following:

Question 6.3.1. What does Poincaré duality look like if we work on a general ring $\mathcal{R}$? What is the analog of (6.2.1) in this setting, i.e., what geometric insights do we get from Poincaré duality?

To answer the above question, we have to add *orientation* of manifolds into the picture. In this section, we will record some covering space theory so that we can define orientation in terms of covering spaces in the next section.

Definition 6.3.2. A*covering map* between topological spaces E and B is a continuous map $p : E \to B$ such that

- for every point $b \in B$, the preimage $p^{-1}(b)$ is discrete,
- every $p \in B$ has a neighborhood V admitting a homeomorphism $p^{-1}(V) \to V \times p^{-1}(b)$.

We call E the *total (covering) space*, B the *base space* and $F \coloneqq p^{-1}(b)$ the *fiber* above $b \in B$.

Example 6.3.3. The projection maps $\sqcup_{i=1}^{n} B \to B$, $\mathbb{S}^n \to \mathbb{RP}^n$, and $o_M \otimes \mathcal{R} \to M$ are covering maps.

Theorem 6.3.4 (Unique path lifting). *Suppose $p : E \to B$ is a covering map and $w : [0,1] \to B$ is a path. Then for any $e \in E$ such that $p(e) = w(0)$ there exists a unique path $\tilde{w} : [0,1] \to E$ such that $\tilde{w}(0) = e$ and $p \circ \tilde{w} = w$.*

Definition 6.3.5. Suppose B is a topological space and $b \in B$. The *fundamental group* $\pi_1(B,b)$ *of* B *at* b is

$$\pi_1(B,b) \coloneqq \{w : [0,1] \to B \mid w(0) = w(1) = b\}/\text{pointed homotopy}$$

where two paths w_1 and w_2 are pointed homotopic if there is a map $h : [0,1] \times [0,1] \to B$ such that

- $h(0, \bullet) = w_1(\bullet)$,
- $h(1, \bullet) = w_2(\bullet)$,
- $h(\bullet, 0) =$ constant path at b,
- $h(\bullet, 1) =$ constant path at b.

Remark 6.3.6. The fundamental group measures the presence of one-dimensional holes in a space. In fact, we have the following theorem stated without proof that relates π_1 and H_1.

Theorem 6.3.7. *Suppose B is a path-connected space. Fix $b \in B$. Then*

$$(\pi_1(B,b))^{ab} \cong H_1(B;\mathbb{Z}),$$

where A^{ab} is the abelianization of the group A.

Remark 6.3.8. There is a right action of $\pi_1(B,b)$ on $F =: p^{-1}(b)$ whenever $p : E \to B$ is a covering map. In fact, consider w to be a representative loop of $[w] \in \pi_1(B,b)$. Then, given $g \in p^{-1}(b)$, lift the loop w to $\tilde{w}$ such that $\tilde{w}(0) = g$ and $\tilde{w}(1) \in F$. Define a right action of $[w]$ on g as

$$g[w] \coloneqq \tilde{w}(1). \tag{6.3.1}$$

One has to check that the definition is indeed an action and it is independent of the choice of representative w.

Definition 6.3.9. A *morphism $\phi : E_1 \to E_2$ of covering spaces E_1 and E_2 of B* is a map that fits into the following commuting diagram

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d] & E_2 \arrow[dl] \\ B. & \end{tikzcd}$$

Remark 6.3.10.

1. The covering spaces of B together with morphisms form a category $\mathbf{Cov}_B$.
2. By $\mathbf{Set}_A$, we mean the category of A-sets, i.e., sets with actions of A. Then there is a functor $\mathbf{Cov}_B \to \mathbf{Set}_{\pi_1(B,b)} = \mathbf{Fun}(\mathbf{B}\pi_1(B,b), \mathbf{Set})$ that sends a pointed covering $(p : E \to B)$ to $p^{-1}(b)$ with an action of $\pi_1(B,b)$. Here, $\mathbf{B}G$ means the one object category corresponding to a group G.

Definition 6.3.11. A space is *semilocally simply connected* if

- it is path connected, and
- for all $b \in B$ and a neighborhood U of b, there exists $V \subset U$ such that $\pi_1(V,b)$ is the trivial group.

Example 6.3.12.

1. $\mathbb{R}^n$ is also semi-locally simply connected.
2. Every path-connected manifold is semi-locally simply connected.

Theorem 6.3.13. *Suppose B is semi-locally simply connected space. Then for all $b \in B$ there is an equivalence of categories $\mathbf{Cov}_B \to \mathbf{Set}_{\pi_1(B,b)}$.*

Proof. See Theorem 31.8 in [Mil16] for details. □

Definition 6.3.14. A *section* $s \in \Gamma(B, E)$ *of a covering map* $p : E \to B$ is a continuous map $s : B \to E$ such that $p \circ s = \mathbb{1}_B$.

Remark 6.3.15. The correspondence $\mathbf{Cov}_B \to \mathbf{Set}_{\pi_1(B,b)}$ allows us to write the space of sections $\Gamma(B, E)$ as

$$\Gamma(B, E) = \{\text{Fixed points of } \pi_1(B, b) \text{ on } p^{-1}(b)\}.$$

6.4 Local orientation

In this section, we will state the Poincaré duality for a general commutative ring $\mathcal{R}$. However, to get a geometric interpretation, we have to take orientation into account. Therefore, most of this section is devoted to defining orientation and formulating topological mechanisms to detect the orientability of manifolds.

Let M be an n-dimensional manifold.

Definition 6.4.1. The *local homology* with coefficients in $\mathcal{R}$ of a manifold M at a point $x \in M$ is $H_*(M, M - \{x\}; \mathcal{R})$.

Remark 6.4.2. In general, $M - \{x\} \subset M$ is not an excision, so we can't always consider the local homology in terms of quotients. However, there is always a long exact sequence

$$\begin{tikzcd}
{} \arrow[r, dotted] & H_q(M - \{x\}; \mathcal{R}) \arrow[r] & H_q(M; \mathcal{R}) \arrow[r] & H_q(M, M - \{x\}; \mathcal{R}) \arrow[dll, rounded corners, "\partial"] \\
& H_{q-1}(M - \{x\}; \mathcal{R}) \arrow[r] & H_{q-1}(M; \mathcal{R}) \arrow[r] & H_{q-1}(M, M - \{x\}; \mathcal{R}) \arrow[r, dotted] & {}
\end{tikzcd}.$$

Suppose U is a neighborhood of x and $U \to \mathbb{R}^n$ is a coordinate map. By the excision $U^{\complement} \subset M - \{x\} \subset M$, we know that $H(M, M - \{x\}) \cong H_*(U, U - \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathcal{R})$. On the other hand, $H_n(\mathbb{R}^n - \{x\}; \mathcal{R}) \cong H_n(\mathbb{R}^n - \{x\}; \mathcal{R}) \cong H_{n-1}(\mathbb{S}^{n-1}; \mathcal{R}) \cong \mathcal{R}$ at degree n and 0 else. Therefore,

$$H_i(M, M - \{x\}; \mathcal{R}) \cong \begin{cases} \mathcal{R} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.4.3. A *local $\mathcal{R}$-orientation of M near $x \in M$* is a choice of generator of $H_n(M, M - \{x\}; \mathcal{R})$ viewed as a rank one $\mathcal{R}$-module.

Remark 6.4.4.
1. When $\mathcal{R} = \mathbb{Z}$ there are two choices of local orientation for any $x \in M$. In particular, choosing a local orientation at x amounts to fixing an orientation (handedness) of a sphere $\mathbb{S}^{n-1}$ in a chart containing x.

2. There is one choice of orientation when $\mathcal{R} = \mathbb{F}_2$ which is part of the reason we bypassed orientation while finding a geometric interpretation to Poincaré inequality.

To be able to state the general Poincaré duality, we need to have a notion of global orientability which boils down to choosing local orientation in a continuous way at every point (cf. Definition 6.4.10).

Definition 6.4.5. Suppose G is a group. A *G-set* is a set A with a map of sets $G \times A \to A$ such that $ea = a$, $(gh)a = g(ha)$ where e is the identity element in G and $g, h \in G$. A *G-torsor* is a G-set A such that for all $a \in A$ the map $g \mapsto ga$ is a bijection $G \to A$.

Remark 6.4.6.

1. A G-torsor is like a group without a choice of identity.
2. Denote $\mathcal{R}^\times$ to be the units in $\mathcal{R}$ under multiplication. Then the local orientation in $H_n(M, M - \{x\}; \mathcal{R})$ is an $\mathcal{R}^\times$ torsor.

Definition 6.4.7. Define

$$o_M := \sqcup_{x \in M} H_n(M, M - \{x\}; \mathbb{Z}) \tag{6.4.1}$$

Note that $o_M \otimes \mathcal{R} = \sqcup_{x \in M} H_n(M, M - \{x\}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{R} = \sqcup_{x \in M} H_n(M, M - \{x\}; \mathcal{R})$.

Definition 6.4.8.

1. For any closed subspace $A \subset M$ and $x \in A$, define

$$j_{A,x} : H_n(M, M - A; \mathcal{R}) \to H_n(M, M - \{x\}; \mathcal{R}) \tag{6.4.2}$$

to be the map induced by the inclusion $(M, M - A) \hookrightarrow (M, M - \{x\})$.

2. For any open subset $U \subset M$ and $\alpha \in H_n(M, M - \overline{U}; \mathcal{R})$, define

$$V_{U,\alpha} := \{ j_{U,x}(\alpha) \mid x \in U \} \subset o_M \otimes \mathcal{R}. \tag{6.4.3}$$

Remark 6.4.9. The collection $\{V_{U,\alpha}\}$ induces a topology on $o_M \otimes \mathcal{R}$ so that the projection map $p : o_M \otimes \mathcal{R} \to M$ is continuous and a covering map.

Definition 6.4.10. An *$\mathcal{R}$-orientation of M* is a section $f \in \Gamma(M, o_M \otimes \mathcal{R})$ such that $f(x)$ is a local orientation for all $x \in M$.

In general, it is hard to check orientability, however, Theorem 6.4.13 gives a way to detect the orientability of a manifold. A general version of the orientability condition is Theorem 6.4.11 which we will deduce by proving Theorem 6.4.15.

For any n dimensional manifold M there is a map

$$j : H_n(M; \mathcal{R}) \to \Gamma(M, o_M \otimes \mathcal{R}) \tag{6.4.4}$$

such that for any $\alpha \in H_n(M; \mathcal{R})$ we define $j(\alpha)(x) \in H_n(M, M - \{x\}; \mathcal{R})$ to be the restriction of $\alpha \in H_n(M; \mathcal{R})$ to $H_n(M, M - \{x\}; \mathcal{R})$ (see the long exact sequence in Remark 6.4.2).

Theorem 6.4.11. *If M is compact then j in (6.4.4) is an isomorphism.*

In particular, a choice of $\mathcal{R}$-orientation corresponds to a generator in the top homology group of M. We denote this generator by $[M]$ and call it the *fundamental class*.

Remark 6.4.12.

1. Recall that a connected manifold is semilocally simply connected. Therefore, Theorem 6.3.13 gives a correspondence $\mathbf{Cov}_M \to \mathbf{Fun}(\mathbf{B}\pi_1(M,x), \mathcal{R}\text{-}\mathbf{mod})$ from which we get a homomorphism

$$\pi_1(M,x) \to \underline{\mathbf{Hom}}_{\mathcal{R}\text{-}\mathbf{mod}}(H_n(M, M-\{x\};\mathcal{R}), H_n(M, M-\{x\};\mathcal{R})) \cong \mathcal{R} \qquad (6.4.5)$$

which dictates the action of $\pi_1(M,x)$. Note that the last equality follows because $H_n(M, M-\{x\};\mathcal{R}) \cong \mathcal{R}$ without a preferred generator.

2. In the light of the Remark 6.3.15, finding sections $\Gamma(M, o_M \otimes \mathcal{R})$ amounts to choosing for each $x \in M$ the fixed points of the action of $\pi_1(M,x)$ on $H_n(M, M-\{x\};\mathcal{R})$ by $\mathcal{R}$-module maps after making choice of generators.

   On the other hand, if we make a choice of a generator, the action of $\pi_1(M,x)$ is given by a homomorphism

$$\pi_1(M,x) \to \mathcal{R}^\times. \qquad (6.4.6)$$

   Since $\pi_1(M,x)$ acts as a module homomorphism, it is clear that the homomorphism (6.4.6) factors through $\mathbb{Z}^\times$, i.e., there is a commutative diagram

$$\begin{tikzcd} \pi_1(M,x) \arrow[r] \arrow[d] & \mathcal{R}^\times \\ \mathbb{Z}^\times. \arrow[ur] & \end{tikzcd}$$

   When the factoring map $\pi_1(M,x) \to \mathbb{Z}^\times = \{\pm 1\}$ is trivial, every element in $\mathcal{R}$ is a fixed point. Therefore, we can choose one element in $\mathcal{R}$ for all x to get a section of $o_M \otimes \mathcal{R}$. Further, Theorem 6.4.11 implies that $H_n(M) = \mathcal{R}$.

   However, when the map $\pi_1(M,b) \to \{\pm 1\}$ is non-trivial, a section of $o_M \otimes \mathcal{R}$ consists of elements of $\mathcal{R}$ fixed by multiplication by ± 1, so they are 2-torsion elements of $\mathcal{R}$ since $r = -r$ implies $2r = 0$. Therefore, Theorem 6.4.11 implies that $H_n(M) = \mathcal{R}[2]$, the set of elements of $r \in \mathcal{R}$ such that $2r = 0$. To summarize the above observation, we have the following theorem.

Theorem 6.4.13. *Suppose M is an n-dimensional compact connected manifold. Then*

$$H_n(M;\mathcal{R}) \cong \begin{cases} \mathcal{R} & \text{if } M \text{ is } \mathcal{R}\text{-orientable}, \\ \mathcal{R}[2] & \text{if } M \text{ is not orientable.} \end{cases}$$

Remark 6.4.14.

1. If $2 = 0$ in $\mathcal{R}$ (for instance $\mathbb{F}_2$) then any connected M is always $\mathcal{R}$-orientable.

2. Suppose M is a connected compact n-dimensional manifold. Then $H_n(M;\mathbb{Z}) \cong \Gamma(M, o_M \otimes \mathcal{R})$ is 0 if M is non-orientable and $\mathbb{Z}$ if it is orientable.

3. Note that $H_2(\mathbb{RP}^2;\mathbb{Z}) \cong 0$, therefore it is not orientable and does not satisfy Poincaré duality. On the other hand, $H_2(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}$, so $\mathbb{T}^2$ satisfies the Poincaré duality.

Theorem 6.4.15. *Let A be a compact subset of an n-dimensional manifold M. Then the natural map $j_A : H_n(M, M-A) \to \Gamma(A, o_M \otimes \mathcal{R})$ is an isomorphism and $H_q(M, M-A) = 0$ for all $q > n$.*

Proof. We will break down the proof into five cases.

Case 1: Suppose $M = \mathbb{R}^n$ and A is convex. The theorem follows because $H_n(\mathbb{R}^n, \mathbb{R}^n - A) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{a\})$ for all $a \in A$.

Case 2: Suppose $M = \mathbb{R}^n$ and A is a finite union of convex sets. We claim that if the theorem is true for A, B and $A \cap B \subset M$, then it is true for $A \cup B \subset M$. In particular, it is true for a finite union of convex sets. Using Mayer–Vietoris, we get

$$\begin{tikzcd}
0 \to H_n(M, M - A \cup B) \arrow[r] \arrow[d] & H_n(M, M-A) \oplus H_n(M, M-B) \arrow[r] \arrow[d] & H_n(M, M - A \cap B) \arrow[d] \\
0 \to \Gamma(A \cup B, o_M \otimes \mathcal{R}) \arrow[r] & \Gamma(A, o_M \otimes \mathcal{R}) \oplus \Gamma(B, o_M \otimes \mathcal{R}) \arrow[r] & \Gamma(A \cap B, o_M \otimes \mathcal{R}).
\end{tikzcd}$$

Then the Five Lemma implies that $j_{A\cup B}$ is an isomorphism.

Case 3: Suppose $M = \mathbb{R}^n$ and A is an arbitrary compact subset. Note that any compact subset $A \subset \mathbb{R}^n$ can be written as $\bigcap A_i$ where $A_0 \subset A_1 \subset A_2$ and each A_i is a finite union of convex subsets of $\mathbb{R}^n$. Further, a section on A extends to a section on A_i for some i.

Case 4: Suppose M is an arbitrary manifold and A is a finite union of compact subsets of $\mathbb{R}^n$. It follows from cases 3 and 2.

Case 5: Finally, suppose M and A are arbitrary. We can write A as a decreasing union of finite copies using the metrizability of $\mathbb{R}^n$. $\square$

Remark 6.4.16. When we set $A = M$ in Theorem 6.4.15 we get Theorem 6.4.11. Further, we get a vanishing theorem that says that $H_q(M;\mathcal{R}) = 0$ for all $q > n$. In particular, the homology groups detect the dimension of a manifold.

Theorem 6.4.17 (Poincaré duality 2.0). *Suppose M is an $\mathcal{R}$-oriented n-dimensional compact manifold. Then there exists a fundamental class $[M] \in H_n(M;\mathcal{R})$ such that for all $p + q = n$ the following composition is a perfect pairing*

$$H^p(M;\mathcal{R}) \otimes_{\mathcal{R}} H^q(M;\mathcal{R}) \xrightarrow{\cup} H^n(M;\mathcal{R}) \xrightarrow{\langle \bullet, [M] \rangle} \mathcal{R}.$$

6.5 Cap product

Question 6.5.1. What is an analog of Kronecker pairing when we replace $\mathbb{F}_2$ by $\mathcal{R}$ in (6.1.1)?

Recall that a pairing $\langle \bullet, \bullet \rangle : S^p(X;\mathcal{R}) \otimes_{\mathcal{R}} S_p(X;\mathcal{R}) \to \mathcal{R}$ induces a Kronecker pairing $H^p(X;\mathcal{R}) \otimes_{\mathcal{R}} H_p(X;\mathcal{R}) \to \mathcal{R}$ which is not always perfect. A case where it is perfect is when $\mathcal{R}$ is a free module and X is a finite CW complex. In any case, a generalization of Kronecker pairing is called the *cap product* $\cap$ and is defined as the composition

$$\cap : S^p(X) \otimes S_n(X) \xrightarrow{1 \otimes A} S^p(X) \otimes S_p(X) \otimes S_{n-p}(X) \xrightarrow{\langle \bullet, \bullet \rangle \otimes 1} S_{n-p}(X),$$

where A is the Alexander Whitney map (see Definition 4.6.4). Then $\cap$ induces a map

$$\cap : H^p(X) \otimes H_n(X) \to H_{n-p}(X). \tag{6.5.1}$$

Remark 6.5.2.

1. Suppose X is a topological space. The map $X \to *$ induces a map $\epsilon : H_*(X;\mathcal{R}) \to \mathcal{R}$ that sends every path component of X to a generator of $\mathcal{R}$. Then for any $b \in H^p(X;\mathcal{R})$ and $x \in H_p(X;\mathcal{R})$ we have
$$\epsilon(b \cap x) = \langle b, x \rangle. \tag{6.5.2}$$

2. The cap product turns H_* into a module over H^*. In fact,
$$(a \cup b) \cap x = a \cap (b \cap x). \tag{6.5.3}$$

3. Suppose $f : X \to Y$ in **Top** and f_* and f^* are the induced maps $f_* : H_*(X) \to H_*(Y)$ and $f^* : H^*(Y) \to H^*(X)$. Then the cap product satisfies the *projection formula*[3]:
$$f_*(f^*(b) \cap x) = b \cap f_*(x). \tag{6.5.4}$$

Theorem 6.5.3 (Poincaré duality 2.0'). *Fix a PID $\mathcal{R}$. Suppose M is a compact, connected, and $\mathcal{R}$-orientable n-manifold. Then there exists a unique fundamental class $[M] \in H_n(M;\mathcal{R}) \cong \mathcal{R}$ (where the isomorphism is $[M] \leftarrow\!\shortmid 1$) such that for any $p + q = n$ the cap product with $[M]$*

$$\bullet \cap [M] : H^p(M;\mathcal{R}) \to H_q(M;\mathcal{R})$$

is an isomorphism of $\mathcal{R}$ modules.

Corollary 6.5.4. *Suppose $p + q = n$. If $H_q(M;\mathcal{R})$ is a free $\mathcal{R}$-module then*

$$H^p(M;\mathcal{R}) \otimes H_q(M;\mathcal{R}) \xrightarrow{\cup} H^n(M;\mathcal{R}) \xrightarrow{\langle \bullet, [M] \rangle} \mathcal{R}$$

is a perfect pairing.

[3]Projection formula is one of the axioms in the six functor formalism used in algebraic geometry

6.6 Čech cohomology

In this section, we will state a more general version of Poincaré duality using Čech cohomology formalism and sketch out the proof. Finally, as an application, we will prove the Jordan curve theorem which roughly states that $\mathbb{S}^1$ divides $\mathbb{R}^2$ into two path-connected components.

Notation: By $U \Subset X$ we mean that U is an open subset of X.

Definition 6.6.1. The *Čech cohomology* of a topological space $X \subset Y$ is defined as the *direct limit*

$$\check{H}^*(X) := \varinjlim_{U \mid X \subset U \Subset Y} H^*(U).$$

Remark 6.6.2.

1. We can think of the direct limit as $\bigoplus_U H^*(U)/\sim$ where for any $U_1 \subset U_2$ we identify $x \sim \iota_*(x)$ for all $x \in H^*(U_2)$. Here $\iota_* : H^*(U_2) \to H^*(U_1)$ is the map induced by the inclusion $\iota : U_1 \hookrightarrow U_2$.

2. $X \subset Y$ then $\check{H}(X)$ is a graded commutative ring.

3. We can define a cap product $\check{H}^p(X) \otimes H_n(X) \to H_{n-p}(X)$.

4. With a little care, we can define relative version of the cap product for $A \subset X$ as

$$\cap : H^p(X) \otimes H_n(X, A) \to H_{n-p}(X, A).$$

   In fact, let $L \subset K$ be closed subspaces of a space X. Then we have a fully relative cap product:

$$\cap : \check{H}^p(K, L) \otimes H_n(X, X - K) \to H_{n-p}(X - L, X - K).$$

   The right hand side lives in a long exact sequence with $H^*(K)$ and $\check{H}^*(L)$.

5. Recall that $H_*(X, X - K)$ is a module over $H^*(U)$. Similarly, there is a commutative diagram

$$\begin{tikzcd}
H^p(U) \otimes H_n(X, X-K) \arrow[r] \arrow[d] & H_{n-p}(X, X-K) \\
H^p(U) \otimes H_n(X, X-K) \arrow[ur] &
\end{tikzcd}$$

   which makes $H_*(X, X - K)$ a module over $\check{H}^*(K)$.

Lemma 6.6.3. *Suppose for every neighborhood $U \Subset Y$ of $X \subset Y$ there exists $V \Subset U$ such that $X \subset V$ and $X \hookrightarrow V$ is a homotopy equivalence. Then $\check{H}^*(X) \cong H^*(X)$.*

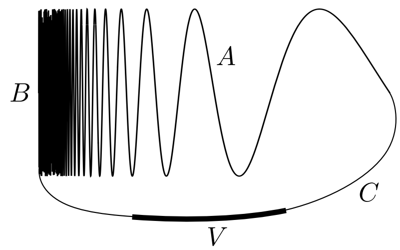


Figure 6.3: Topologist's sine curve

Remark 6.6.4. The isomorophism in Lemma 6.6.3 does not hold for general X. Consider the union of topologist's sine curve and an arc. More precisely, let A be the graph of $\sin(2\pi/x)$ for $x \in (0,1)$, $B := 0 \times [-1,1]$ and C is an arc from $(0,-1)$ to $(1,0)$. Define $X := A \cup B \cup C$, see Figure 6.3. Then $\check{H}^*(X) \cong H^*(\mathbb{S}^1)$. On the other hand, let $V \subset C$ be a small segment and $U \subset X$ be the neighborhood of $X - V$. Then using Meyer–Vietoris, we get a sequence

$$0 \to H_1(X) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X)$$

which implies that $H^1(X) = 0$.

Now we are ready to state Poinaré duality.

Theorem 6.6.5 (Poincaré duality 3.0). *Let $\mathcal{R}$ be any commutative ring and $L \subset K$ a pair of compact subsets of an n-dimensional manifold M. Assume we are given a fundamental class $[M]_K \in H_n(M, M-K;\mathcal{R})$ that restricts to a generator of $H_n(M, M-\{k\};\mathcal{R})$ for every $k \in K$. Then for any $p+q=n$, the map*

$$\bullet \cap [M]_K : \check{H}^p(K, L;\mathcal{R}) \to H_q(M-L, M-K;\mathcal{R})$$

is an isomorphism.

Proof sketch. **Case 1:** Start with $M = \mathbb{R}^n$ and convex compact subsets $K, L \subset \mathbb{R}^n$.

Case 2: Assume $M = \mathbb{R}^n$ and K and L are finite unions of convex compact subsets. Use Meyer Vietoris and the fact that intersection of convex sets remain convex.

Case 3: Consider $M = \mathbb{R}^n$ and K and L arbitrary. Use the fact that every compact subset of $\mathbb{R}^n$ is intersection of a family of $A_1 \supset A_2 \supset \ldots$ of infinite unions of convex sets.

Case 4: Prove for the case when M is a manifold, K and L are homeomorphic to compact subsets of $\mathbb{R}^n \subset M$.

Case 5: Finally, prove the general case. □

Here are some applications of Theorem 6.6.5

Corollary 6.6.6. *Let M be an n-dimensional manifold and K a compact subset. An $\mathcal{R}$-orientation along K determines an isomorphism $\check{H}^{n-q}(K;\mathcal{R}) \to H_q(M; M-K;\mathcal{R})$.*

Corollary 6.6.7. *Suppose K is a compact subset of $\mathbb{R}^n$. Then $\check{H}^{n-p}(K) \cong H_p(\mathbb{R}^n, \mathbb{R}^n - K)$.*

Theorem 6.6.8 (Jordan curve theorem). *Let K be a compact subspace of $\mathbb{R}^2$ such that $\check{H}^*(K) \cong H^*(\mathbb{S}^1)$. Then $\mathbb{R}^2 - K$ has two path components.*

Proof. We have an isomorphism $\check{H}^1(K) \cong H_1(\mathbb{R}^2, \mathbb{R}^2 - K)$. Further, we have an exact sequence

$$H_1(\mathbb{R}^2 - K) \to H_1(\mathbb{R}^2) \to H_1(\mathbb{R}^2, \mathbb{R}^2 - K) \to H_0(\mathbb{R}^2 - K) \to H_0(\mathbb{R}^2).$$

Note that the second term is 0, the third is $\mathbb{Z}$ by assumption and the last term is $\mathbb{Z}$. Therefore, the fourth term is $\mathbb{Z} \oplus \mathbb{Z}$. □

Remark 6.6.9.

1. Knot theory generalizes the Jordan curve theorems and studies low dimensional "curves."
2. The Jordan curve theorem fails in $\mathbb{R}^3$, see Alexander horned sphere.

Appendix A

Further reading

In this chapter,[1] we will highlight some of the possible directions that one could pursue after reading these notes:

- category theory,
- homotopy theory,
- algebraic topology,
- homotopy groups of spheres, and
- homological algebra.

A.1 Category theory

Previously, we defined pushouts and products. They are special cases of limits and colimits which are tools to study *universal properties*. More generally, we use adjoints, limits, and representables to understand the universal properties of objects, see [Mac88]. Further, we defined a category to have objects and morphisms. We can generalize the concept so that there are maps between morphisms.

Definition A.1.1. A 2-*category* is a category with objects, morphism and 2-morphism (a morphism of morphisms).

Example A.1.2. 1. $\mathbf{Top}_2$ is a 2-category where objects are topological spaces, morphisms are continuous maps and 2-morphisms are homotopy between maps.

[1] This chapter should be taken as a pinch of salt. I don't have any expertise in these subjects to cross-check all the facts in case I incorrectly transcribed the lectures. However, the buzzwords should be enough for an enthusiastic reader to go forward. I might edit this section as I keep building more background. However, I have decided to keep it here so that I can come back later and make sense of what I have written down.

2. **Cat**$_2$ is 2-category where objects are categories, morphisms are functors and 2-morphisms are natural transformations.

3. **chAb** is 2-category where objects are chain complexes, morphisms are chain maps and 2-morphisms are chain homotopies.

Similarly, we can define n-category. In fact, we have the following definition:

Definition A.1.3. An ∞-*category* is a category with objects, morphisms, n-morphisms for all $n \in \mathbb{N}$.

Example A.1.4. The ∞-category $\mathcal{H}$ of homotopy types has the following data:

- objects: topological space (geometrical realization of semisimplicial sets),
- morphisms: continuous maps,
- 2-morhpisms: if $f : X \to Y$ and $g : X \to Y$ are morphisms then a 2-morphism is a map $h : X \times [0, 1] \to Y$ with $h(\bullet, 0) = f$ and $h(\bullet, 1) = g$,
- 3-morphisms between h_1 and h_2 is a map $H : X \times [0, 1] \times [0, 1] \to Y$ and so on.

The study of $\mathcal{H}$ is called *homotopy theory.*

A.2 Homotopy theory

Homotopy theory is the study of the equality of spaces. For instance, in Figure A.1, a and b are equal in one way, and we can identify them as a point. On the other hand, c and d are equal in two ways that are not equal. Once we identify c and d via one way, we still have a loop left. In contrast, e and f are equal and there are infinite ways in which e and f are equal. In fact, there is equality in the ways in which they are equal. Therefore, by identifying e and f and the ways in which they are equal, we get a point.

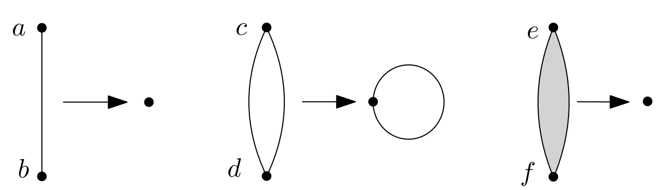


Figure A.1: Equalities

One of the objects widely studied in homotopy theory is the set $\pi_a\mathbb{S}^b$ of maps $\mathbb{S}^a \to \mathbb{S}^b$ up to homotopy (equality) and its cardinality.

Recall that a pushout of the diagram in **Top**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

is $B \sqcup C/(f(a) = g(a))$ which can be thought of as $A \sqcup B \sqcup C/(a = f(a), a = g(a))$, see Figure A.2. Analogously, a pushout in $\mathcal{H}$ of the above diagram is $(A \times [0,1]) \sqcup B \sqcup C/((a,0) = f(a), (a,1) = g(a))$. However, homotopy pushout is not alwyas equivalent to the classical pushout. For instance, when $C = *$ the the classical pushout is $B/\operatorname{Im}(f)$. But the homotopy pushout is a cone, see Figure A.3.

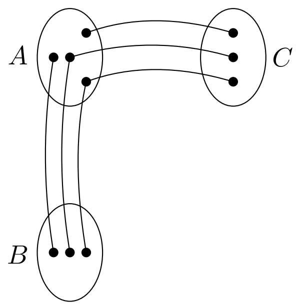


Figure A.2: Classical pushout

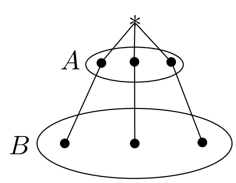


Figure A.3: Homotopy pushout

However, we have the following theorem:

Theorem A.2.1. $H_n(B, A) = H_n(P_{\mathcal{H}})$ *where* $P_{\mathcal{H}}$ *is the homotopy pushout.*

To learn more about ∞-categories and homotopy theory, see Kerodon or take 18.906.

A.3 Algebraic topology

Another theme of our course has been algebraic topology which is the use of homotopy theory to study geometry. The main question in the field as we posed in Question 1.1.1 can be simplified to the following question:

Question A.3.1. Can we classify all n-dimensional compact manifold up to homeomorphism (diffeomorphisms if the manifolds are required to be smooth)?

The following is a more concrete version:

Question A.3.2 (Generalized Poincaré conjecture)**.** Suppose M is a compact, connected n-dimensional manifold such that

- $\pi_1(M, m) = 0$ where $m \in M$, and
- $H_*(M; \mathbb{Z}) = H_*(\mathbb{S}^n; \mathbb{Z})$.

Must M be homeomorphic (diffeomorphic) to $\mathbb{S}^n$?

Up to homeomorphism, the answer is yes. For $n = 1, 2$, it was known in the 1900s. Smale (1960) proved it for all $n \geq 5$, see [Sma07]. Freedman (1982) proved it for $n = 4$, see [Fre82]. Perelman (2003) proved it for $n = 3$, see [Per02], [Per03a] and [Per03b]. All of the aforementioned authors were awarded Fields medal for their work (although Perelman rejected it).

Up to diffeomorphism, the answer is yes for $n = 3$ (Perelman). In contrast, Milnor (1959) constructed a counter example (exotic spheres) for $n = 7$, see [Mil56]. The case when $n = 4$ is still open while the answer is yes if $n = 5$ and 6. In all dimension $n \geq 5$ where $n \not\equiv 7 \mod 8$, Kervaire and Milnor (1963) gave a formula for the number of exotic spheres in terms of various $|\pi_a S^b|$, see [KM63]. The formula was extended to (n even) $n/2 \equiv 7 \mod 8$ by Hill–Hopkins–Ravenel (2009) in [HHR16] except for $n = 126$ (which is still unknown). See here for a more references.

In Question A.3.2, M has non-trivial groups at two degrees. We could ask the same question when the groups are non-trivial at three degrees. For $n \neq 1, 4, 8$, the question was settled by Wall, Milnor in the 60s. Stolz (1983) proved for $n = 1 \mod 8$ and Prof. Hahn and et. al (2019) proved for $n = 0 \mod 4$. In any case, an important object in all works turns out to be $|\pi_a \mathbb{S}^b|$.

We can make $\pi_a \mathbb{S}^b$ an abelian group $a, b \geq 1$ with a group law addition. Consider two maps $f : \mathbb{S}^a \to \mathbb{S}^b$ and $g : \mathbb{S}^a \to \mathbb{S}^b$. Then we define $f + g : \mathbb{S}^a \to \mathbb{S}^b$ as follows:

- We ensure (using homotopic maps) that the image of the south pole of $\mathbb{S}^a$ under f is same as that of the north pole of $\mathbb{S}^a$ under g.
- Pinch the equator to get $\mathbb{S}^a \to \mathbb{S}^a \vee \mathbb{S}^a$. Then f sends the top copy of $\mathbb{S}^a$ to $\mathbb{S}^b$ while g sends the bottom copy to $\mathbb{S}^b$, see Figure A.4.

Example A.3.3. $\pi_{14}(\mathbb{S}^4) \cong \mathbb{Z}/120\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Click here to see more examples.

In 18.906, we will prove the following theorems:

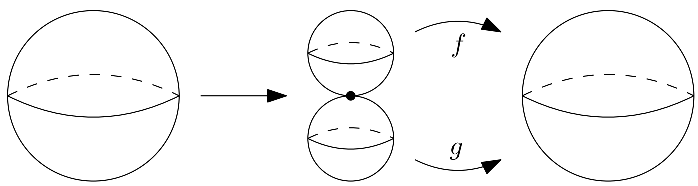


Figure A.4: $f+g$

Theorem A.3.4. *$\pi_a\mathbb{S}^b$ is a finite group if $a>b$ and $a=2b-1$. In contrast, if $a<b$ then $\pi_a\mathbb{S}^b=0$. Finally, if $a=b$ the maps are in bijection (determined by degree) with $\mathbb{Z}$.*

Theorem A.3.5 (Freundenthal). *For all $b\geq k+2$, $\pi_{b+k}\mathbb{S}^b$ is independent of b.*

The "stable" group is denoted by $\pi_k S$. In manifold theory, we mostly care about stable groups $\pi_k\mathbb{S}^b$.

A.4 Homotopy group of spheres

In the previous section, we introduced $\pi_a\mathbb{S}^b$. Here we will give more examples and present the current state of affairs.

Example A.4.1. The following computations were individual theorems in the 50s.

$$\pi_3\mathbb{S}^2\cong\mathbb{Z}.$$
$$\pi_4\mathbb{S}^3\cong\mathbb{Z}/2\mathbb{Z}$$
$$\pi_5\mathbb{S}^4\cong\mathbb{Z}/2\mathbb{Z}.$$

Remark A.4.2. Consider a map $f:\mathbb{S}^a\to\mathbb{S}^b$ where $a\geq b$. Then $H_*(f;\mathbb{Z})=0$.

Definition A.4.3. Consider a map $f:\mathbb{S}^a\to\mathbb{S}^b$. We say that f has *$H\mathbb{F}_p$-Adams filtration at least k* if it can be factored as a composite

$$X_0=S^a\xrightarrow{f_1}X_1\xrightarrow{f_2}X_2\xrightarrow{f_3}\dots\xrightarrow{f_k}X_k=S^k$$

where $H_*(f_i;\mathbb{F}_p)$ is the zero map for each i.

Intuitively, a map f with high Adams filtration is invisible to $\mod p$ homologies.

Recall that $\pi_k\mathbb{S}$ is a finite abelian group which can be written as a direct sum of cyclic groups of order p for various primes p. For each p, we can organize the p-component of $\pi_k\mathbb{S}$ by $H\mathbb{F}_p$ Adams filtration and study them using Adams chart. Only the first 40 of them were computed as of 2016. We have now pictures of the 90th group. The groups of higher Adams filtration are v_1 periodic maps. Click here to see a picture of Adams chart. To prove that there is no polynomial time to compute these finite group is open. The height of figures is related to the Riemann zeta function at negative integers. Other related questions is to know if there is a second gap in the Adams chart. The first one has slope 1/3. On the other hand, the second one if it exists is believed to have a slope of 1/6.

A.5 Homological algebra

Definition A.5.1. An *extraordinary cohomology theory* is a functor $E : \mathbf{Top} \to \mathbf{Ab}$ satisfying every Eilenberg–Steenrod axiom except the dimension axiom. In particular $E_*(point)$ need not be concentrated at degree 0.

Example A.5.2. K-Theory (KO) is the cohomology theory of vector bundles. Here,

$$KO_*(p) \cong \begin{cases} \mathbb{Z} & \deg 0, \\ \mathbb{Z}/2 & \deg 1, \\ \mathbb{Z}/2 & \deg 2, \\ 0 & \deg 3, \\ \mathbb{Z} & \deg 4, \\ 0 & \deg 5, \\ 0 & \deg 6, \\ 0 & \deg 7, \end{cases}$$

where the groups are 8-periodic. The phenomenon is called Bott periodicity. In general, KO sees exactly v_1 periodic groups. In the past, people have used K-theory to solve problems in geometry regarding Hopf invariant. On a different note, recall that we proved in homeworks that $\mathbb{R}^3$ does not have a linear multiplication structure. In fact, using KO theory we can prove that there is no non-linear multiplication on $\mathbb{R}^3$.

Other homology theories inspired form geometry are KO, MU, MO, MSpin theories. The idea is to see what homotopy groups of spheres we can see using different theories. The trade-off is computability. We also have designer homology theories that uses algebra instead of geometry. More generally, we study cohomology theories not necessarily derived from geometry or algebra. The main question is:

Question A.5.3. Can we classify the cohomological functors E?

The question turns out to be related to higher algebra which refers to algebra done in $\mathcal{H}$. Here, $\mathcal{H}$ is to **Set** as E_∞ (defined below) is to **Ab**. In $\mathcal{H}$, we keep track of **Set** as well as equalities.

Definition A.5.4. An $\mathbb{E}_1$-*space* is an $X \in \mathcal{H}$ with a "group operator" $X \times X \to X$ such that for every triple of points we have $(x_1x_2)x_3 = x_1(x_2x_3)$ (up to homotopy).

Remark A.5.5. $\mathbb{E}_1$ ring is to associative ring and $\mathbb{E}_\infty$ is to commutative ring.

The following theorem gives an answer to the Question (A.5.3).

Theorem A.5.6 (Brown–May). *Homology theories are in bijection with $\mathbb{E}_\infty$ spaces.*

Example A.5.7. We can think of a classical abelian group A as a discrete set and study $H_*(-, A)$.

We can push the theory and do (spectral) algebraic geometry in $\mathbb{E}_\infty$ spaces and ask the following questions:

Question A.5.8. How do classical algebraic geometry theorems generalize to higher world?

It turns out that there are some phenomena that are unclear in classical algebra but are visible in higher algebra.

Definition A.5.9. An *elliptic cohomology theory* is an $\mathbb{E}_\infty$ ring corresponding to the global sections of spectral elliptic curve.

In classical theory, a moduli stack of elliptic curve is not affine, so there is no universal elliptic curve. However, in spectral algebraic geometry, there is a universal curve called the topological modular form (TMF) that controls all other elliptic curves. We use topological because

$$TMF_*(pt) \otimes_{\mathbb{Z}} \mathbb{C} = \text{ classical ring of modular forms}$$

TMF theory is a 576-periodic theory. A main question that is still open in TMF theory is

Question A.5.10. Does TMF have a geometric interpretation?

String theorists already seem to know the answer intuitively. In particular, TMF is related to the index of Dirac operators on loop spaces. But nobody has been able to solve this problem rigorously.

Another direction is to study chromatic homotopy theory. Here, every $\mathbb{E}_\infty$ ring has a *chromatic height*. For instance, we have $H_*(0; \mathbb{Z})$ at 0 height, KO_* at height 1, TMF at height 2 and Lubin–Tate theory at height n. A big open question in this area is the following question:

Question A.5.11. What number theory is connected to chromatic heights greater than 2?

On a different direction we have the following theorem:

Theorem A.5.12 (Chromatic convergence theorem). *Every element in $\pi_a \mathbb{S}^b$ is detected at some finite chromatic height.*

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