Moser's iteration scheme

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Last updated: September 19, 2021

Abstract

I wrote this note to grasp the idea behind Moser's proof of Hölder continuity of weak solutions of second order elliptic partial differential equation that can be written in divergence form but not to produce any original work.

1 Introduction

We are interested in the regularity of weak solutions of the equation:

$$\boldsymbol{\nabla} \cdot (a \boldsymbol{\nabla} u) = 0 \tag{1.1}$$

where a is a symmetric positive definite matrix that satisfies the ellipticity condition:

$$\lambda \mathbb{1} \leq a \leq \Lambda \mathbb{1} \tag{1.2}$$

for some $\lambda, \Lambda > 0$. The equation (1.1) appears frequently in variational problems in math and natural sciences. In fact, the problem of existence and regularity of the minimizer of variational problem was Hilbert's 19th and 20th problems. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Hilbert was concerned when $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ is a smooth function that satisfies for some constants $\lambda, \Lambda > 0$ and for all $\xi, \eta \in \mathbb{R}^n$

$$\lambda |\xi|^2 \leq \langle \mathrm{Hess}_{\mathcal{L}}(\eta)\xi, \xi \rangle \leq \Lambda |\xi|^2$$

where $\mathrm{Hess}_{\mathcal{L}}$ is the Hessian of $\mathcal{L}$. In physics literature, $\mathcal{L}$ is known as Lagrangian. Consider u to be the minimizer of the functional (action in physics literature) $S(v) := \int_\Omega \mathcal{L}(\boldsymbol{\nabla} v)$ where we take v such that the integral on the right makes sense. In other words, for all v in that space $S(u) \leq S(v)$ which implies the Euler–Lagrange equation:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \mathcal{L}(\boldsymbol{\nabla} u)) = 0. \tag{1.3}$$

In fact, consider a test function φ and define $I(t) = S(u + t\varphi) = \int_\Omega \mathcal{L}(\boldsymbol{\nabla} u + t \boldsymbol{\nabla} \varphi)$. Since u is a minimizer, I achieves its minimum at 0. Therefore, $I'(0) = 0$. Meanwhile,

$$I'(t) = \int_\Omega \langle \boldsymbol{\nabla} \mathcal{L}(\boldsymbol{\nabla} u + t \boldsymbol{\nabla} \varphi), \boldsymbol{\nabla} \varphi \rangle = - \int_\Omega \boldsymbol{\nabla} \cdot (\mathcal{L}(\boldsymbol{\nabla} u + t \boldsymbol{\nabla} \varphi))\varphi.$$

Since φ is arbitrary, the Euler–Lagrange equation follows. Now formally differentiating (1.3), we get

$$\boldsymbol{\nabla} \cdot (\mathrm{Hess}_{\mathcal{L}}\, \boldsymbol{\nabla} u) = 0$$

which is in the divergence form (1.1).

Before stating the theorem, let's record some definitions and notations. We say that a locally integrable function $v \in L^1_{loc}(\Omega)$ is α^{th} weak derivative of $u \in L^1_{loc}(\Omega)$ written $D^\alpha u = v$ if

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v\varphi$$

for all test functions $\varphi \in C_c^\infty(\Omega)$. By $W^{k,p}(\Omega)$ for all $1 \le p \le \infty$, we mean the Sobolev space consisting of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex $|\alpha| \le k$, $D^\alpha u$ exists in a weak sense and lies in $L^p(\Omega)$. A function $u \in W^{1,2}(\Omega)$ is a weak solution of (1.1) if for all non-negative test functions φ

$$\int_\Omega \langle a \boldsymbol{\nabla} u, \boldsymbol{\nabla} \varphi \rangle = 0,$$

and it is a weak subsolution (supersolution) if the equality is replaced by $\le$ ($\ge$).

In this note, we will follow Moser's idea to prove the regularity of weak solutions to (1.1) [Mos61].

Theorem 1.1 (Hölder continuity). *A weak solution $u \in W^{1,2}(\Omega)$ of the equation (1.1) belongs to $C^{0,\alpha}(\Omega)$ where the Hölder exponent α depends on n, λ and Λ only.*

Around 1950's De Giorgi proved the regularity theorem for elliptic partial differential equations (PDE) [DG57] using class of functions satisfying reverse Poincaré inequalities. Meanwhile, Nash independently proved the regularity for parabolic PDEs [Nas60] using heat kernel estimates. In contrast, Moser used Harnack's inequality to prove the regularity.

Theorem 1.2 (Harnack's inequality). *For every weak solution of (1.1) $u \in W^{1,2}(\Omega)$ there exists a positive constant $c = c(n, \lambda, \Lambda)$ such that for every ball $B(x, r) \subset \Omega$,*

$$\sup_{B(x,\frac{r}{2})} u \le c \inf_{B(x,\frac{r}{2})} u. \tag{1.4}$$

Proof of Theorem 1.1. Note that Theorem 1.1 immediately follows from Theorem 1.2. In fact, define

$$M(r) \coloneqq \sup_{B(x,r)} u \qquad\qquad m(r) \coloneqq \inf_{B(x,r)} u$$

which means essential sup and essential inf if necessary. Define $g(r) \coloneqq M(r) - m(r)$. We claim that for all $r' \le \frac{r}{2}$

$$g(r') \le c_0 \left(\frac{r'}{r}\right)^\alpha g(r) \tag{1.5}$$

for some α to be determined and a constant c_0. It implies

$$u(x) - u(y) \le \sup_{B(x,r')} u - \inf_{B(x,r')} u = g(r') \le \frac{c}{r^\alpha} g(r)|y-x|^\alpha$$

for $|x-y| \le r'$. Now covering Ω by finite number of balls, Hölder continuity follows.

Thus, it suffices to prove (1.5). To this end, note that $M(r') - u$ and $u - m(r')$ are non-negative solutions. Using Theorem 1.2, we get

$$M(r) - m(\frac{r}{2}) \le \sup_{B(x,\frac{r}{2})} (M(r) - u) \le c \inf_{B(x,\frac{r}{2})} (M(r) - u) = c\Big(M(r) - M(\frac{r}{2})\Big). \tag{1.6}$$

Similarly,

$$M(\frac{r}{2}) - m(r) \le c\Big(m(\frac{r}{2}) - m(r)\Big). \tag{1.7}$$

Adding (1.6) and (1.7), we get $g(\frac{r}{2}) \le \nu g(r)$ where $\nu = \frac{c-1}{c+1}$. Using induction, we get

$$g(\frac{r}{2^n}) \le \nu^n g(r)$$

for all $n \in \mathbb{Z}^+$. For general r', we can find n such that $\frac{r}{2^{n+1}} < r' \le \frac{r}{2^n}$. Fix α such that $\nu \le \frac{1}{2^\alpha}$. Meanwhile, the monotonicity of g implies

$$g(r') \le g(\frac{r}{2^n}) \le \nu^n g(r) \le \frac{1}{2^{n\alpha}} g(r) \le 2^\alpha \left(\frac{r'}{r}\right)^\alpha g(r).$$

□

2 Proof of Harnack's inequality

In this section, we will prove Theorem 1.2. The key idea is to bound both sides of (1.4) in terms of $\Phi(p,r) \coloneqq \left(\fint_{B(y,r)} |u|^p\right)^{\frac{1}{p}}$ where $\fint_A f$ is the average $\frac{\int f}{|A|}$ and use the monotonicity of $\Phi(p,r)$ with respect to p. In fact, for $p \le q$, using Hölder's inequality

$$\left(\fint_{B(y,r)} |u|^p\right)^{\frac{1}{p}} \le \frac{1}{|B(y,r)|^{\frac{1}{p}}} \left(\int_{B(y,r)} (|u|^p)^{\frac{q}{p}}\right)^{\frac{q}{p}p} \left(\int_{B(y,r)} dx\right)^{\frac{1}{p}-\frac{1}{q}} = \left(\fint_{B(y,r)} |u|^q\right)^{\frac{1}{q}}.$$

Note that Theorem 1.2 follows by combining the monotonicity of $\Phi(p,r)$ and the following two theorems.

Theorem 2.1. *Suppose $u \in W^{1,2}(\Omega)$ is a weak solution of the equation (1.1). Then $u \in L^\infty_{loc}(\Omega)$. In particular, for every ball $B(y,r) \subset \Omega$ and all $p > 0$*

$$\sup_{B(y,\frac{r}{2})} |u| \le c\Phi(p,r) \tag{2.1}$$

where $c > 0$ depends only on n, λ and Λ.

Theorem 2.2. *Suppose $u \in W^{1,2}(\Omega)$ is a weak solution of the equation (1.1). Then there exists $q, c > 0$ both depending only on n, λ and Λ such that for every ball $B(r, 2r) \subset \Omega$,*

$$\inf_{B(y,\frac{r}{2})} u \geq c\Phi(q, r). \tag{2.2}$$

To prove Theorem 2.1 and 2.2, we have to use a reverse Poincaré inequality (cf. Theorem 2.6) and an iteration argument of Moser along with three inequalities that we state without proof. For the first two see [Eva10]. The third one is an application of Doob's maximal inequality for martinagles, see [Koi21].

Lemma 2.3 (Poincaré inequality). *Fix $1 \leq p < \infty$. There exists a constant c depending on n and p such that*

$$\int_{B(y,r)} |u - u_{B(y,r)}|^p \leq cr^p \int_{B(y,r)} |\boldsymbol{\nabla} u|^p \tag{2.3}$$

for every $u \in W^{1,p}(\Omega)$ and every ball $B(y, r) \subset \Omega$, where $u_{B(y,r)} = ⨍_{B(y,r)} u$.

Lemma 2.4 (Sobolev inequality). *Fix $1 \leq p < n$ and define $p^* = \frac{np}{n-p}$. There exists a constant c depending on n and p such that*

$$\left(\int_\Omega |u|^{p^*}\right)^{\frac{1}{p^*}} \leq c\left(\int_\Omega |\boldsymbol{\nabla} u|^p\right)^{\frac{1}{p}} \tag{2.4}$$

for all u in the closure of test functions in $W^{1,p}(\Omega)$.

A function $u \in L^1_{loc}(\Omega)$ has bounded mean oscillation and is written $u \in BMO(\Omega)$ if

$$\|u\|_{BMO,1} \coloneqq \sup_{B(y,2r)\subset\Omega} ⨍_{B(y,r)} |u - u_{B(y,r)}| < \infty.$$

Lemma 2.5 (John–Nirenberg inequality). *For every $u \in BMO(\Omega)$, there exist constants c_1 and c_2 depending only on n and $\|u\|_{BMO,1}$, such that for every ball $B(y, 2r) \subset \Omega$:*

$$⨍_{B(y,r)} e^{c_1|v - v_{B(y,r)}|} \leq c_2. \tag{2.5}$$

2.1 Proof of Theorem 2.1

Unless otherwise stated, $u \in W^{1,2}(\Omega)$ is a positive weak solution of (1.1). We can work with $u^+ \coloneqq \max(u, 0)$ and $u^- \coloneqq -\min(u, 0)$ and combine the proof for $u = u^+ - u^-$.

Lemma 2.6. *For any $\alpha \geq 0$ and a test function φ, there exists $c > 0$ depending on λ and Λ*

$$\int_\Omega |u|^\alpha |\boldsymbol{\nabla} u|^2 \varphi^2 \leq c \int_\Omega |u|^{\alpha+2} |\boldsymbol{\nabla}\varphi|^2 \tag{2.6}$$

whenever the integral on right side makes sense.

Proof. Fix a test function φ. Then

$$\int_\Omega |u|^\alpha |\nabla u|^2 \varphi^2 = \int_{t\in(0,\infty)} \alpha t^{\alpha-1} \int_{u>t} |\nabla u|^2 \varphi^2.$$

Meanwhile,

$$\int_{u>t} |\nabla u|^2 \varphi^2 \leq \frac{1}{\lambda} \int_{u>t} \langle a\nabla u, \nabla u\rangle \varphi^2 = \frac{1}{\lambda} \int_{u>t} \big(-2\langle a\nabla u, \nabla \varphi\rangle \varphi u + \langle a \nabla u, \nabla\big(u\varphi^2\big)\rangle\big).$$

Note that the second term in the last integral vanishes because u is a weak solution and $u\varphi^2$ is in the closure of the test function in $W^{1,2}(\Omega)$, so the claim follows from density. For the first term, we use Cauchy–Schwarz to get

$$-\langle a\nabla u, \nabla \varphi\rangle \leq \langle a\nabla u, \nabla u\rangle^{\frac{1}{2}} \langle a\nabla \varphi, \nabla\varphi\rangle^{\frac{1}{2}}$$

which together with a different Cauchy–Schwarz implies

$$\begin{aligned}\frac{1}{\lambda}\int_{u>t} -2\langle a\nabla u, \nabla\varphi\rangle\varphi u &\leq \frac{1}{\lambda}\left(4\int_{u>t}\langle a\nabla u, \nabla u\rangle \varphi^2\right)^{\frac{1}{2}} \left(\int_{u>t}\langle a\nabla\varphi, \nabla\varphi\rangle u^2\right)^{\frac{1}{2}} \\ &\leq \frac{2\Lambda}{\lambda}\left(\int_{u>t} |\nabla u|^2\varphi^2\right)^{\frac{1}{2}} \left(\int_{u>t} |\nabla \varphi|^2 u^2\right)^{\frac{1}{2}}\end{aligned}$$

whence

$$\int_{u>t} |\nabla u|^2 \varphi^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{u>t} |\nabla\varphi|^2 u^2.$$

Therefore,

$$\int_\Omega |u|^\alpha |\nabla u|^2\varphi^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{t\in(0,\infty)} \alpha t^{\alpha-1} \int_{u>t} |\nabla\varphi|^2 u^2 = \frac{4\Lambda^2}{\lambda^2}\int_\Omega |u|^{\alpha+2}|\nabla\varphi|^2.$$

□

Lemma 2.7. *For every $p \geq 1$, $u \in L^p_{loc}(\Omega)$. Further, for every $\alpha \geq 0$, every ball $B(y,r) \subset \Omega$ and all $0 < r' < r$, there exists a constant $c > 0$ depending only on n, λ and Λ such that*

$$\left(\int_{B(y,r')} |u|^{(\alpha+2)\kappa}\right)^{\frac{1}{(\alpha+2)\kappa}} \leq \frac{c^{\frac{1}{\alpha+2}}(\alpha+2)^{\frac{2}{\alpha+2}}}{(r-r')^{\frac{2}{\alpha+2}}} \left(\int_{B(y,r)} |u|^{\alpha+2}\right)^{\frac{1}{\alpha+2}} \tag{2.7}$$

where $2\kappa := 2^*$.

Proof. It suffices to prove

$$\left(\int_{B(y,r')} |u|^{(\alpha+2)\kappa}\varphi^{2\kappa}\right)^{\frac{1}{\kappa}} \leq c(\alpha+2)\int_\Omega |u|^{\alpha+2}|\nabla\varphi|^2 \tag{2.8}$$

where we can choose φ to be a test function supported on $B(y,r)$ with values 1 in $B(y,r')$ and $|\boldsymbol{\nabla}\varphi| \leq \frac{1}{r-r'}$.

To this end, we use Sobolev inequality to $v = |u|^{\frac{\alpha}{2}} u\varphi$ to get

$$\left(\int_{B(y,r')} |u|^{(\alpha+2)\kappa}\varphi^{2\kappa}\right)^{\frac{1}{2\kappa}} = \left(\int_{B(y,r')} |v|^{2^*}\right)^{\frac{1}{2^*}} \leq c\left(\int_{B(y,r')} |\boldsymbol{\nabla} v|^2\right)^{\frac{1}{2}}.$$

On the other hand,

$$\boldsymbol{\nabla} v = \left(\frac{\alpha}{2}+1\right)|u|^{\frac{\alpha}{2}}\varphi\boldsymbol{\nabla} u + |u|^{\frac{\alpha}{2}}u\boldsymbol{\nabla}\varphi \leq \left(\frac{\alpha}{2}+1\right)\left(|u|^{\frac{\alpha}{2}}\varphi\boldsymbol{\nabla} u + |u|^{\frac{\alpha}{2}}u\boldsymbol{\nabla}\varphi\right).$$

Now using $(e+f)^2 \leq 2e^2+2f^2$, we can bound $|\boldsymbol{\nabla} v|^2$ by $\left(\frac{\alpha}{2}+1\right)^2(|u|^{\alpha}\varphi^2|\boldsymbol{\nabla} u|^2 + |u|^{\alpha+2}|\boldsymbol{\nabla}\varphi|^2)$. Therefore, using Lemma 2.6, we get

$$\int_{\Omega} |\boldsymbol{\nabla} v|^2 \leq c(\alpha+2)^2 \int_{\Omega} |\boldsymbol{\nabla} u|^{\alpha+2}|\boldsymbol{\nabla}\varphi|^2$$

which implies (2.8). □

Lemma 2.8. *For every ball $B(y,r) \subset \Omega$ and $0 < \sigma < 1$:*

$$\sup_{B(y,\sigma r)} u \leq \frac{c}{(1-\sigma)^{\frac{n}{2}}}\Phi(2,r) \tag{2.9}$$

where $c > 0$ depends on n, λ and Λ.

Proof. Here we iterate (2.7) for specific choices of parameters. In particular, fix $B(y,r) \subset \Omega$. Define $\alpha_i := 2\kappa^i - 2$ and $r_i := \sigma r + \sigma^{i+1} r$ for $i \in \mathbb{N}$. Using $r = r_i$, $r' = r_{i+1}$ and $\alpha = \alpha_i$ for $i \in \mathbb{N}$ in (2.7) and writing $W_{i+1} := \left(\int_{B_{y,r_i}} |u|^{2\kappa^i}\right)^{\frac{1}{2\kappa^i}}$, we get

$$W_{i+1} \leq \frac{c^{\frac{1}{2\kappa^i}}(2\kappa^i)^{\frac{1}{\kappa^i}}}{(\sigma^{i+1}r)^{\frac{1}{\kappa^i}}(1-\sigma)^{\frac{1}{\kappa^i}}} W_i$$

whence

$$W_{i+1} \leq \prod_{j=0}^{i} \frac{c^{\frac{1}{2\kappa^j}}(2\kappa^j)^{\frac{1}{\kappa^j}}}{(\sigma^{j+1}r)^{\frac{1}{\kappa^j}}(1-\sigma)^{\frac{1}{\kappa^j}}} W_0.$$

Note that $\prod a_j$ converges if and only if $\sum \log a_j$ converges. In our case, the logarthmic test guarantees the convergence since κ^j grows faster than j. Meanwhile, $\prod \frac{1}{(1-\sigma)^{\frac{1}{\kappa^j}}} = \frac{1}{(1-\sigma)^{\sum \frac{1}{\kappa^j}}} = \frac{1}{(1-\sigma)^{\frac{n}{2}}}$. Letting i to ∞ and taking into account the volume of balls $B(y,\frac{r}{2})$ and $B(y,r)$, we get

$$\sup_{B(y,\sigma r)} u \leq \frac{c}{(1-\sigma)^{\frac{n}{2}}}\Phi(2,r).$$

□

Proof of Theorem 2.1. Note that Lemma 2.8 already implies the bound

$$\sup_{B(y,\frac{r}{2})} u \leq c\Phi(p,r) \tag{2.10}$$

for $p \geq 2$ by monotonicity of $\Phi(p,r)$ with respect to p. Therefore, it suffices to prove the bound for $0 < p < 2$ which again follows from Lemma 2.8 by iteration argument.

In fact, fix $0 < \sigma < 1$ and $B(y,r) \subset \Omega$. Define $\sigma_i := 1 - \frac{1-\sigma}{2^i}$. For $i \in \mathbb{N}$, we replace σ by $\frac{\sigma_i}{\sigma_{i+1}}$ and r by $\sigma_{i+1}r$ in Lemma 2.8 to get

$$\sup_{B(y,\sigma_i r)} u \leq \frac{c}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{\frac{n}{2}}} \Phi(2, \sigma_{i+1}r)$$

Using positivity of u and $2 - p > 0$, we get

$$\begin{aligned}
\Phi(2,\sigma_{i+1}r) &= \left(\fint_{B(y,\sigma_{i+1}r)} u^{p+2-p}\right)^{\frac{1}{2}} \\
&\leq \left(\fint_{B(y,\sigma_{i+1}r)} u^{p}\right)^{\frac{1}{2}} \left(\sup_{B_(y,\sigma_{i+1}r)}\right)^{\frac{2-p}{2}} \\
&\leq \left(\fint_{B(y,r)} u^{p}\right)^{\frac{1}{2}} \left(\sup_{B(y,\sigma_{i+1}r)} u\right)^{\frac{2-p}{2}}.
\end{aligned}$$

Denote $M_i := M(\sigma_i r) = \sup_{B(y,\sigma_i r)} u$. Then we have the following recursion formula:

$$M_i \leq \frac{c}{\left(1 - \frac{\sigma_i}{\sigma_{i+1}}\right)^{\frac{n}{2}}} \left(\fint_{B(y,r)} u^p\right)^{\frac{1}{2}} M_{i+1}^{\frac{2-p}{2}} \tag{2.11}$$

whence

$$\sup_{B(y,\sigma r)} u = M_0 \leq \frac{c^{\sum_{j=0}^{i}\left(\frac{2-p}{2}\right)^j}}{\prod_{k=1}^{i}\left(1 - \frac{\sigma_k}{\sigma_{k+1}}\right)^{\frac{n}{2}\left(\frac{2-p}{2}\right)^k}} \left(\fint_{B(y,r)} u^p\right)^{\frac{1}{2}\sum_{j=0}^{i}\left(\frac{2-p}{2}\right)^j} M_i^{\left(\frac{2-p}{2}\right)^i}.$$

As $i \to \infty$ we know that $M_i \to \sup_{B(y,r)} u$ which is essentially bounded, therefore the last factor goes to 1. On the other hand, the integral converges to $\Phi(p,r)$. The factor with c converges to $c^{\frac{2}{p}}$. Meanwhile, $1 - \frac{\sigma_k}{\sigma_{k+1}} = \frac{1-\sigma}{2^{k+1}\sigma_{k+1}}$ implies the numerator of the denominator of the first factor contributes $(1-\sigma)^{\frac{n}{p}}$ to the product. Further, the exponential decay $\left(\frac{2-p}{2}\right)^k$ guarantees that $\prod_k \left(2^{k+1}\sigma_{k+1}\right)^{\frac{n}{2}\left(\frac{2-p}{2}\right)^k}$ converges. Therefore, letting $\sigma = \frac{1}{2}$, we get

$$\sup_{B(y,\frac{r}{2})} u \leq c(n,p,\lambda,\Lambda)\Phi(p,r).$$

□

2.2 Proof of Theorem 2.2

In this section, we will assume that u is strictly positive solution of (1.1). In fact, we can work with $u+\epsilon$ for $\epsilon > 0$ and take $\epsilon \to 0$. In the first, case, we will prove that $\log u \in BMO(\Omega)$ (the property being independent of ϵ) and use John–Nirenberg inequality to prove Theorem 2.2.

Lemma 2.9. *For any $q > 0$, there exists $c > 0$ depending on q, n, λ and Λ such that for every ball $B(y,r) \subset \Omega$:*

$$\inf_{B(r,\frac{r}{2})} u \geq c\Phi(-q,r). \tag{2.12}$$

Proof. Note that Theorem 2.1 works for subsolutions. Then (2.12) follows by replacing u in Theorem 2.1 by $\frac{1}{u}$. We just have to show that $\frac{1}{u}$ is a subsolution. In fact, $\frac{1}{u}$ belongs to $W^{1,2}(\Omega)$. Further, for any test function $\varphi \geq 0$

$$\begin{aligned}\int_\Omega \langle a\boldsymbol{\nabla}\frac{1}{u}, \boldsymbol{\nabla}\varphi\rangle &= -\int_\Omega \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}\varphi\rangle\frac{1}{u^2}\\ &= -2\int_\Omega \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}u\rangle\frac{\varphi}{u^3} - \int_\Omega \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}\Big(\frac{\varphi}{u^2}\Big)\rangle \leq 0\end{aligned}$$

where the non-positivity follows because the first term in the third line is negative by the ellipticity of a and the second term vanishes because u is a solution and $\frac{\varphi}{u^2}$ is a non-negative test function. ∎

Define $v := \log u$. The lemma that follows provides an estimate for the oscialltion of v and will be crucial in proving that $v \in BMO(\Omega)$.

Lemma 2.10. *For every ball $B(y,2r) \subset \Omega$,*

$$\int_{B(y,r)} |\boldsymbol{\nabla}v|^2 \leq cr^{n-2}$$

where $c > 0$ depends only on n, λ and Λ.

Proof. Fix a cutoff function φ to be determined. Then

$$\int_\Omega |\boldsymbol{\nabla}v|^2\varphi^2 = \int_\Omega |\boldsymbol{\nabla}u|^2\frac{\varphi^2}{u^2} \leq \frac{1}{\lambda}\int_\Omega \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}u\rangle\frac{\varphi^2}{u^2} = \frac{1}{\lambda}\Big(\int_\Omega 2\langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}\varphi\rangle\frac{\varphi}{u} - \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}\Big(\frac{\varphi^2}{u}\Big)\rangle\Big).$$

Note that the second term in the last integral vanishes because $\eta = \frac{\varphi^2}{u}$ is a test function and u is a solution. On the other hand, Cauchy–Schwarz implies

$$\langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}\varphi\rangle \leq \langle a\boldsymbol{\nabla}u, \boldsymbol{\nabla}u\rangle^{1/2}\langle a\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\varphi\rangle^{1/2}$$

which together with another Cauchy–Schwarz implies

$$\begin{aligned}\frac{1}{\lambda}\int_\Omega 2\langle a\boldsymbol{\nabla} u, \boldsymbol{\nabla}\varphi\rangle\frac{\varphi}{u} &\le \frac{1}{\lambda}\left(4\int_\Omega \langle a\boldsymbol{\nabla} u, \boldsymbol{\nabla} u\rangle\frac{\varphi^2}{u^2}\right)^{\frac{1}{2}}\left(\int_\Omega \langle a\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\varphi\rangle\right)^{\frac{1}{2}}\\ &\le \frac{2\Lambda}{\lambda}\left(\int_\Omega |\boldsymbol{\nabla} v|^2\varphi^2\right)^{\frac{1}{2}}\left(\int_\Omega |\boldsymbol{\nabla}\varphi|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\int_\Omega |\boldsymbol{\nabla} v|^2\varphi^2 \le \frac{4\Lambda^2}{\lambda^2}\int_\Omega |\boldsymbol{\nabla}\varphi|^2.$$

Now fixing φ to have a support on $B(y,2r)$ with value 1 on $B(y,r)$ and $|\boldsymbol{\nabla}\varphi| \le \frac{1}{r}$, we get the desired bound. □

Proof of Theorem 2.2. Note that Poincaré inequality and Lemma 2.10 imply for every ball $B(y,2r)\subset\Omega$

$$⨍_{B(y,r)}|v - v_{B(y,r)}|^2 \le cr^{2-n}\int_{B(y,r)}|\boldsymbol{\nabla} v|^2 \le c(n,\lambda,\Lambda)$$

whence $v \in BMO(\Omega)$ which by John–Nirenberg inequality implies there exists constant $c_1, c_2 > 0$ depending only on n and λ and Λ such that $⨍_{B(y,r)} e^{c_1|v-v_{B(y,r)}|} \le c_2$.

On the other hand,

$$⨍_{B_(y,r)} u^{c_1}⨍_{B_(y,r)} u^{-c_1} = ⨍_{B(y,r)} e^{c_1(v-v_{B(y,r)})}⨍_{B(y,r)} e^{c_1(v_{B(y,r)}-v)} \le \left(⨍_{B(y,r)} e^{c_1|v-v_{B(y,r)}|}\right)^2 \le c_2^2.$$

Now using (2.12) to bound the left most term from below, we get

$$\inf_{B(y,\frac{r}{2})} u \ge c\Phi(c_1, r).$$

□

Acknowledgement

I would like to thank Dan Stroock for his constructive comments on the first draft and for motivating me to go through the proof of the iteration scheme meanwhile also teaching Nash's approach to Hölder continuity.

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